

Spacetimes with semantics II (supplement)\*

# On the scaling of functional spaces, from smart cities to cloud computing

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## Abstract

The study of spacetime, and its role in understanding functional systems has received little attention in information science. Recent work, on the origin of universal scaling in cities and biological systems, provides an intriguing insight into the functional use of space, and its measurable effects. Cities are large information systems, with many similarities to other technological infrastructures, so the results shed new light indirectly on the scaling the expected behaviour of smart pervasive infrastructures and the communities that make use of them.

Using promise theory, I derive and extend the scaling laws for cities to expose what may be extrapolated to technological systems. From the promise model, I propose an explanation for some anomalous exponents in the original work, and discuss what changes may be expected due to technological advancement.

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\*These notes are a continuation my series on semantic spacetimes. This document is inspired by the studies of coarse-grained universal scaling in cities [1–3], and a comparison with models developed over the past decade or two on information systems, e.g. [4, 5]. As IT systems grow in scale, is natural to expect a bridge between the behaviours of cities, software networks, and other functionally ‘smart’ spaces, and one hopes for a better understanding of pervasive information technology in social contexts. Work on this, from the low level viewpoint, has already begun in [6, 7]. I want to show how these views relate.

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## 1 Introduction

Two recent works have shed light on the role of infrastructure in the scaling of functional systems. The observation of universal, scale free behaviours, both in observed data, and in a 'mean field' description of cities [1, 2], and the observation, in biology, of metabolic scaling laws for organisms, following a century of observation [8]. Both of these were explained from the behaviours of their internal networks.

Cities are an example of collaborative community networks, where humans and technology mix within a semantically (i.e. functionally) rich space, equipped with infrastructure. One may ask how cities differ from apparently similar communities across different scales, including tribes, collectives, companies, software installations, and even countries. Understanding the dominant processes that make these shared environments smart, creative, and productive, is a worthy investment, given how 21st century life relies on them so much for its success and survival.

In this note, I review and build on Bettencourt's model of cities [1], and discuss its implications for pervasive information technology (IT). By building on the lessons of cities, I hope to foster a better understanding of a broader range of functional systems, especially in information technology, while trying to preserve the simplicity of Bettencourt's arguments. I begin by summarizing an interpretation that lays

the foundations for a more microscopic formulation of the model, using promise theory [9]. The latter may be used to relate outcomes to intentions and mechanisms, in a way that respects the idea of scaling.

## 2 Bettencourt's model of scaling in cities summarized

This section is a review (and trivial generalization) of the city scaling arguments, and data used by Bettencourt and collaborators at the Santa Fe Institute (SFI) [1–3], in a form suitable for comparison with other work. In section 3, I propose a deeper justification for the model, with some embellishments. Some interpretations may be my own.

### 2.1 Scaling phenomenology

Measurable attributes, of finite dynamical systems, typically scale in proportion to some measure of their size. Size may refer to the number of agents  $N$  (persons) in the system, or per unit volume  $V$  of the system, and  $N$  and  $V$  may or may not be related. This is a point of view that is a basic tool of analysis in physical systems. For cities,  $N$  is used as the scaling variable. Across an ensemble of many systems of different size, the measurements one obtains may scale in three broad ways (see figure 1):

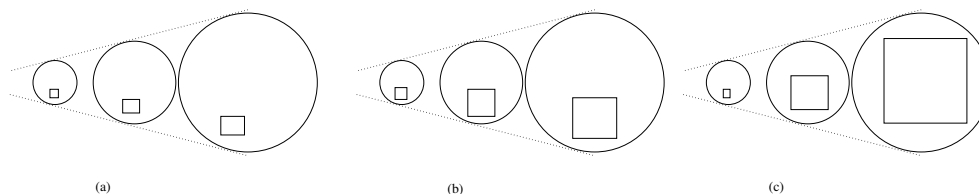


Figure 1: Scaling of a square quantity, relative to the circular system size (a) sublinear, (b) linear, (c) superlinear. If an cost is superlinear, it applies a braking force on city size.

1. Sublinear scaling of quantities  $q$  of the infrastructure machinery  $q \propto N^{\beta < 1}$ . This indicates economies of scale, because, as the system size grows, the cost becomes relatively cheaper. In cities, it is observed to apply to the transport networks that animate the system (arterial systems, roads, cables, petrol stations, etc).
2. Linear scaling, simply proportional to the number  $q \propto N^{\beta=1}$ . In cities, this seems to apply to direct consumption of goods and resources per capita (jobs, houses, water consumption, etc).
3. Superlinear yields of produce or ‘output’, where  $q \propto N^{\beta > 1}$ , which is driven by interactivity between the parts of the system and its consequences (wages, rents, patents, crime, disease, GDP). If a process rate is superlinear, then the corresponding time for the process to run will be sublinear, and vice versa.

It is of particular interest when these patterns seem to apply across such a broad range of scales. This suggests some emergent *universality*, whose origin and mechanism is worth understanding.

As part of a protracted project to uncover the behaviours of cities or urban metropolitan districts, Bettencourt has proposed a mean field model to explain observed scaling behaviour of certain economic measures, making only elementary assumptions about the processes within [1]. The model predicts the main features of the data, by assuming a dynamical universality, but seems to fall short of describing the full range of observed scaling exponents  $\beta$ . Empirical data from [2] revealed sublinear, linear, and

superlinear scaling behaviour in the variety of accessible data. The data came initially from mostly American cities (see table 1), and have since been demonstrated in European cities in [10]. All cities, thus far, have a more or less comparable level of technological development, and thus fit plausibly into a statistical ensemble.

SUBLINEAR	Linear	SUPERLINEAR
Fuel sales	Housing	New patents
Fuelling stations	Employment	Inventors
Length of cables	Power consumption	R&D employment
Road surface	Water consumption	Other creative employment
		Disease (AIDS)
		Crime

Table 1: Examples of city outputs with sub- and superlinear scaling per capita, reported in [2].

Cities are not just dynamical systems, they also exhibit complex semantics, or purposeful, intentional, patterned behaviours. In the physics of inanimate systems, the markers of semantics are comprised of only a few simple labels; charges, force laws, and interaction graphs. These are constant over time, and accepted as universal ‘physical law’. However, in human functional systems, the range of interactions, and their assumed meanings, is much richer, and may depend of time and context. Coarse graining, by creating a mean field model, is a standard physical approach which eliminates the types, labels, and other semantics of networks, and exposes the universality of scaling phenomena. However, this simplicity is a trade-off: semantics are also what sustains the arrangement and composition of functional processes, at a deeper level, and could lead to actionable predictions.

In these notes, I show how semantics provide some additional structure and constraints on dynamics, and how both short and long range interactions may be distinguished through functional dependency. This leads to a possible explanation for the discrepancy between data and predictions of superlinear scaling exponents in [1, 2] using a slightly more detailed model than [1], based on promise theory [9].

## 2.2 Definitions

Consider a city, in the coarsest approximation, as a bounded homogenous mass, sustained by external supplies and internally generated wealth. This is analogous to a model of gravity and pressure versus volume, whose equilibrium defines an average city size. Bettencourt likens the arrangement to a star, which gravitates together from the benefits of city infrastructure, and expands through the accretion of new inhabitants.

The city is populated by  $N$  inhabitants, which I will call ‘agents’, which and may include machines, animals, etc., or any proxies for human intent that lead to economic output. The data on cities, used in the comparison, are based on human population numbers, so  $N$  will refer to the people, in the first approximation. The agents are distributed within a volume  $V$ , in  $D$  dimensions. Cities are more or less two dimensional ( $D = 2$ ), in spite of high rise regions, because the more or less 1 dimensional networks connecting them lie mainly in a plane.

## 2.3 Outline of the approach

- Let the number of agents in the network, or city community, be:

$$N = N_I + N_0 \quad (1)$$

where  $N_0$  is a partial dead-weight population that is not interacting with the city infrastructure (as user or provider)<sup>1</sup>, and the functional networks generating the yields are agents from the  $N_I$  population. Bettencourt does not distinguish between  $N_I$  and  $N_0$ , but it is useful to track this distinction throughout the scaling argument.

A dependence on  $N$  can mean two different things, in the formulae: an average value of a static population, across an ensemble of different cities, or a dynamically growing value of  $N$  within a single city, over the course of its evolution. The interpretation of the results is sensitive to this distinction, requiring some care. If universality were completely true, they would be the same.

- A city is a volume  $V$  of agents, accreted from a wider region, with finite compressibility, by virtue of some average space requirements per person, and a pressure a (non-detailed) balance condition, which matches the output shared amongst interactions to the resources that can be fed to the connected agents.
- An estimate is made of the minimum fraction of the volume to be infrastructure  $V_I$  that could connect the city's agents together into a virtual mesh.
- The city infrastructure is assumed to be 'sparsely' utilized, at equilibrium, in the sense that the technology on which ensemble cities are based can absorb fluctuations in space and time, without significant contention or expansion<sup>2</sup>, else it would choke to a standstill. This is a technical (statistical) property, which is necessary to achieve economies of scale, through spacetime multiplexing<sup>3</sup>. A city may appear to be busy, even crowded at certain moments, but 'sparse' means that it could technically get a lot more crowded, on average, without collapsing from congestion.
- Various functional output yields  $Y$ , of the city may be calculated in a form suitable for ensemble comparison, to determine their scaling with  $N$ . Some outputs stem from individual agents, and some are from the interactions between them.

## 2.4 Explicit and implicit assumptions behind a mean field model

To make a mean field explanation plausible, over a range of scales and circumstances, some assumptions are required. The mean field model should not be confused with a detailed model of physical processes. It is an effective representation that emphasizes universal aspects of behaviour. Expanding on what is stated in [1]:

- The ability to form an ensemble over multiple cities, assumes that when similar outputs are promised, they are made by the same kinds of agents, with the same basic capabilities [13]. If one city has a technology that makes the same output twice as productive, this will not necessarily respect the ensemble's universality assumption, leading to anomalies. The latter assumption suggests that there might be difficulties comparing third world cities, primitive societies, or fully automated production facilities with more homogenized average candidates.
- The distribution of agents within a city cannot be predicted or described by a mean field model. I'll return to this issue in section 3. The agents involved in a specific output 'yield'  $Y$  might be concentrated into regions, however, there may also be multiple regions with the same role, adding

<sup>1</sup>The dead weight could be green spaces, but more likely slum dwellings such as those that dominate Baltimore. Hot-beds for crime and shadow economy. This merely interferes with city output as far as the world is concerned.

<sup>2</sup>In Feynman's words, there is 'plenty of room at the bottom'. Fluctuations in network utilization generally exhibit long-tailed behaviour [11, 12], so we must be far away from the point of collapse, in spite of superficial appearances like traffic jams.

<sup>3</sup>Arrival processes like Poisson and Lévy distributed events have the property that a convolution of multiple flows is form invariant.

up to the total partial volume  $V_Y$  used in the expressions. All are lumped into a single equivalent volume  $V_Y$  for the purpose of total comparison (see figure 2)..

- A key assumption is that infrastructure filaments take up only a small or ‘sparse’ fraction of volume of the city  $V_I \ll V$ , but can cope with all of its load requirements. This sparse use of resources, with the city volume, will be important to justifying the superlinear scaling.
- In this rendition of the model, I add one trivial extension to [1]: the infrastructure network connects a fraction of  $N_I \leq N$  of the total population  $N$  together, leaving some residual number  $N_0$  of city dwellers unconnected, or non-participatory. This allows us to track the possible impact of a non-productive mass.
- The infrastructure network is not a mesh network itself, but it delivers sufficient capacity and interconnectivity to allow a virtual mesh of coverage, i.e. any agent can reach any other agent with equal average cost.
- The maximum output of the city community is assumed to follow Metcalfe’s law, which estimates the productivity of a network proportionally to the maximum number of links that can be made [14]. This has been criticized theoretically (for a sample see [15, 16]). However, recently this conjecture has received empirical support from social media studies [17]<sup>4</sup>. I will derive this result, and its limitations, from promise theory in section 3.6.3, and also show that this cannot be the only measure of productivity to reproduce the data.
- Let us measure both value and cost in the dimensions of ‘money’  $[M]$ . In [1], the author uses power (energy per unit time) as the proposed currency; however, money might be easier to grasp for many readers.

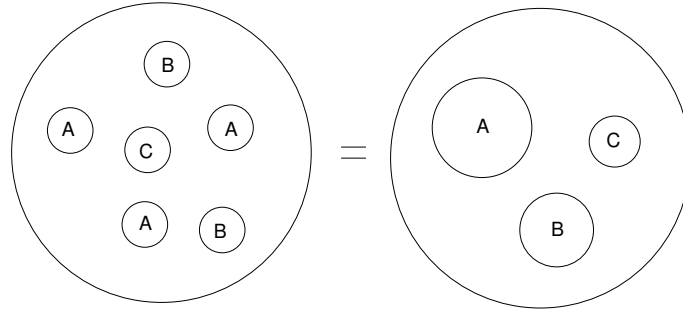


Figure 2: The yield agency model is a pudding model, in which different yields may be spread about inside the city bounds, but they may all be lumped into a single equivalent volume for the purpose of scaling the ingredients.

Regarding the economic output of the city, the model is quite simple.

- The city’s activities yield outputs, labelled by different  $Y$ .
- These come from groups of agents that interact and collaborate in an unspecified way. The group occupies an aggregate volumes  $V_Y$ , for each  $Y$  (see figure 2). These patches are virtually contiguous even when they are physically distributed, and so they may overlap in space.

<sup>4</sup>Metcalfe originally assumed that value creation would be proportional to  $N^2$  while costs grew proportional to  $N$ . The study [17] indicates that both grow quadratically with network vertex count, though these ideas are still disputed [15, 16].

- By assuming that  $V_Y$  is a fraction of the  $N_I^2$  interaction mesh, one implicitly allows the members of a yield producer to be located anywhere within the equal-cost network.

The economic output due to a process  $Y$  may be written approximately as:

$$Y = g_Y \frac{v_Y^{\text{coop}}}{V_I} N_I^2, \quad (2)$$

where  $g_Y$  has units of money  $[M]$ , and  $N_I$  is assumed dimensionless. In many cases we can assume that  $N_I \simeq N$ , so that the entire city is active (no dead weight or free riders), but there is no need to make this identification yet.

## 2.5 The fraction of volume that is sparse infrastructure network

In the volume of the city, network links are counted using a continuum approximation in terms of volumes, and fractions of volumes. This is an unfamiliar step in computer science, but it plays an important role in deriving the scaling laws, and this case can offer valuable lessons. The minimum size of the infrastructure network can be estimated by squeezing the total sparse volume into a narrow, approximately one dimensional pipeline, with a small cross section. This is only plausible if the network utilization is really sparse, since then the total interaction can be compressed into the lower dimensional network, by multiplexing. The average distance between agents inside the city (in  $D$  dimensions) is

$$d = \left( \frac{V}{N} \right)^{\frac{1}{D}}. \quad (3)$$

An infrastructure network has the topology of a graph, in the mathematical sense, but it may also have sufficient structure to pervade space<sup>5</sup>. It is embedded in a real world volume, and needs to reach the agents distributed within. If the agents cluster around the network, the system will remain largely one dimensional; however, if the network penetrates the space homogeneously (either by wiring or by the movement of inhabitants who use it), then (again, in the spirit of generality) the effect of this ‘space filling’, or fractal invasion<sup>6</sup>, may be captured by an effective (Hausdorff) dimension  $H < D$ , so that we may write the order of magnitude estimate for the infrastructure volume:

$$V_I \geq g_I \left( \frac{V}{N} \right)^{\frac{H}{D}} L^{D-H} N_I, \quad (4)$$

where  $g_I < 1$  is a dimensionless constant that indicates the fraction of nodes in  $N_I$  spanned by the particular infrastructure being considered.  $L$  is some fixed scale with the dimensions of length  $[L]$ , so that

$$[V] = [L]^D. \quad (5)$$

and  $L^D \ll V_I \ll V$ . In other words, the volume is the effective average linear volume swept out by a fixed cross section  $L^{D-H}$ , as it feeds into the  $N_I$  nodes connected by the infrastructure<sup>7</sup>. This has the

<sup>5</sup>This was an important argument in deriving the biological scaling laws [8].

<sup>6</sup>The model cannot formally distinguish between the intricacy of the infrastructure itself and the movement of agents around it, but it makes sense to assume that it is the motion of people and mobile agents that is complex, rather than the system of roads and wires of the city. I’m grateful to Luis Bettencourt for this comment.

<sup>7</sup>It is well known that the scaling of ad hoc communications networks, where agents are distributed randomly is like  $\sqrt{N}$ . This is easily understood from the spatial geometry: mobile phones occupy some approximately two dimensional area, so the diameter is of order  $N^{\frac{1}{2}}$ ; alternatively, they have average separation  $d \simeq V/\sqrt{N}$ , so the distance across the group is of the order  $Nd \simeq \sqrt{N}$ . The linearity of the process gets mixed up with the geometry of the embedding space.

schematic form of  $N_I$  queues that are serialized paths of dimension  $V^{1/D}$ . For  $H = 0$  the nodes are unconnected, for  $H = 1$  roads are serial or linear, and for  $D > H > 1$ , the roads or channels have an effective fractal ‘thickness’, from a coarse-grain perspective (see fig. 3). It turns out that we only need to look at  $H = 1$ , as it is serialization rather than physical dimension that is important.

The assumption that we can squash a volume  $V$  into a smaller linearized volume  $V_I$  is the key process, or universal mechanism for comparison between cities. It expresses what we mean by a ‘sparsely utilized’ infrastructure, i.e. the gas of inhabitants is somewhat compressible. One thinks first of physical channels, such as roads, cables, and transportation costs etc; however, any serial stream of work could constitute work process from a number of agents within the volume  $V$ . Thus we write the infrastructure volume as

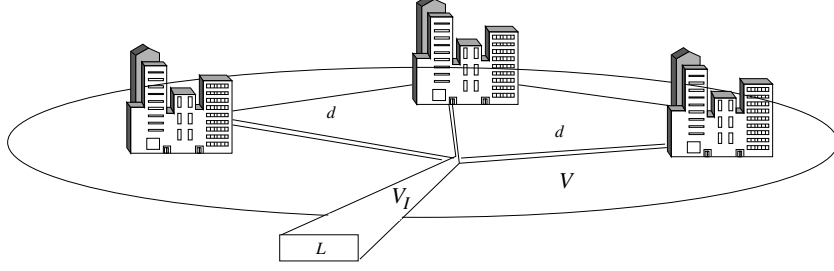


Figure 3: The volume of the infrastructure sparse network is negligible compared to the ball of the city.

approximately (rewriting (4)):

$$V_I \geq g_I V^{\frac{H}{D}} L^{D-H} N_I N^{-\frac{H}{D}}, \quad (6)$$

representing  $N_I$  serial queues of supporting services, being fed from a  $D$  dimensional region. The  $V^{1/D}$  scaling represents a serialization of the work from across the homogeneous region.

## 2.6 Inflating the community: sustainable city volume at steady state

Although the city population and volume are presumed to accrete, as people come, attracted by the promises of the city, a given population has to be sustainable. The total population and volume of the city must satisfy an equation of state, that explains the average balance of these payments. The scaling of these payments is the same, but in reverse, so the balance ends up only as a sign to the coefficients. From the structure, there must be three main parts:

- **Agent sustenance:** the existence constraint on agent survival, individually, represents a separate concern that does not play directly into the scaling of the city (every agent for itself). It is represented, implicitly, through the assumption of constant  $N$ . This has two aspects: the attractive force that brings people to the city, and the resources that feed and supply the balance of payments. Few cities are self-sufficient within their bounds. These aspects remain formally unexplained in [1], and thus do not play a detailed role in the data described.
- **Work output sustenance:** output yields  $Y$ , produced by the various agents of the city, whether individuals or factories, are assumed to make the city cash flow positive, together with the input of external resources, making that the city is economically viable. This feedback is not modelled directly, only assumed by the positive signs of the coupling coefficients along links, and the assumption of steady state. Thus, cities might be profitable, or borrowing money to reach this steady state. There is no way to capture these factors in the mean field approximation.



- **City volume sustenance:** what we can model is the supply of resources must balance the outputs of the city.

$$\text{Running cost} \leq \text{Resource supply in} - \text{Transport cost.} \quad (7)$$

The running and transport costs are assumed to be low, else the agents would not be able to sustain their existence. The costs are in the links and in the linearized supply through the infrastructure.

The feeding of resources (perhaps from outside the city) through the linearized infrastructure (RHS) powers interactions within the partial city volumes, over many activities, (LHS), and effectively inflates the volume of the city by placing a scale  $V$  over which resources must be transported. This yields a simple (non-detailed) balance condition:

$$\begin{aligned} \text{Cost of interaction links} &\leq \text{Supply through infrastructure} \\ g_Y \frac{v_Y}{V} N_I^2 &\leq c_Y V^{\frac{H}{D}} N, \end{aligned} \quad (8)$$

where we assume that work cost is proportional to the distance travelled (analogous to  $W = F \cdot dl$ ), and  $c_Y$  is the cost per unit length of path transport along the infrastructure (dimensions  $[M][L]^{-H}$ ). The positive coupling  $g_Y$  includes the balance of payments to keep the city viable. Rearranging this inequality gives the implied constraint on the sustainable volume:

$$V \geq a \left( \frac{N_I^2}{N} \right)^{\frac{D}{D+H}} \quad (9)$$

where  $a = (g_Y v_Y^{\text{maintain}} / C_Y)^{\frac{D}{D+H}}$ .

## 2.7 Scaling of spatial infrastructure and output yields

Substituting the steady state volume of the city into the expression for the infrastructure volume, the model now predicts the three kinds of scaling from the introduction:

1. **Sublinear.** The scaling of infrastructure itself in terms of the volume, we get:

$$V_I = g_I a^{H/D} \left( \frac{N_I^2}{N} \right)^{-\frac{2HD+H^2}{D(D+H)}} \quad (10)$$

setting  $D = 2$  and  $H = 1$  into (6),

$$V_I \simeq \left( \frac{N_I^2}{N} \right)^{\frac{5}{6}}. \quad (11)$$

For pervasive  $N \simeq N_I$ , this yields

$$V_I \simeq N^{\frac{5}{6}}, \quad (12)$$

giving the sublinear scaling observed by [1]. When the infrastructure cost is basically absent, this scales like ‘every man for himself’, like an ideal gas of non-interacting agents.

2. **Linear.** For individual agent consumption, the scaling is trivially linear, by assumption, both inputs (consumption  $C$ ) and outputs  $O$ .

$$C_- = e_- N \quad (13)$$

$$C_+ = e_+ N \quad (14)$$

where the dimensions  $[e] = [M]$  are of money.

3. **Superlinear.** The positive economic yield of a process in the city, due to interactions may be written as a fraction of the possible  $N_I^2$  output that can be channelled through the volume  $V_I$ , with a different constant of proportionality for each output:

$$Y_Y^+ = G_Y \frac{N_I^2}{V_I}, \quad (15)$$

where  $G_Y$  is assumed positive, absorbing the costs of interaction, and from (6) we have an expression for the infrastructure volume, up to invariant unknowns, which are simply constants that do not depend on the state variables  $N$  or  $V$ . So, substituting (10) into (15),

$$Y_Y^+ = (g_I G_Y L^{H-D}) V^{-\frac{H}{D}} N_I \frac{N_I^H}{V^{\frac{H}{D}}}, \quad (16)$$

and substituting for volume in (9), since it also depends on  $N_I$ :

$$Y_Y^+ = \left( g_I G_Y L^{H-D} a^{-\frac{HD}{D(H+D)}} \right) N^{\frac{2HD+H^2}{D(H+D)}} N_I^{\frac{D^2-HD}{D(H+D)}}, \quad (17)$$

If we substitute  $D = 2$ , and  $H = 1$ , this scales as

$$Y_Y^+ \simeq N_I^{\frac{7}{6}} \left( 1 + \frac{N_0}{N_I} \right)^{\frac{5}{6}}. \quad (18)$$

And if we further assume that the infrastructure network is pervasive, so that  $N \simeq N_I$ , then

$$Y_Y^+ \simeq N_I^{\frac{7}{6}} \simeq N^{\frac{7}{6}}. \quad (19)$$

This reproduces the superlinear scaling identified theoretically in [1], and this result matches about half of the superlinear scaling data quite well. Other data show significantly higher values for the scaling. If the infrastructure network is insignificantly small  $N_I \ll N_0$ , then there would be binomial corrections to a  $1/N$  scaling. The ‘dead population’ reduces the power law slightly (in binomial corrections), so we could expect processes, for which most of the city cannot contribute, to scale below the  $7/6 = 1.16$ th power, thus this cannot explain the higher scaling exponent. An enhanced explanation is proposed in section 3.7.

## 2.8 Remarks about the calculation

In spite of a smattering of city related narrative, this calculation is based on a very simple and universal argument about resources exchanged between a  $D$  dimensional volume and a one dimensional supply network. There must be sufficient spacetime volume associated with ensemble-standard infrastructure to accommodate growth in  $N_I$ , or an increase in density of the users, without saturating it. Multiplexing in space and time, is they key to this.

- Measures, relating to the output of self-sufficient agents, scale linearly, as one would expect. If we double the number of suppliers, there is twice as much availability.
- Shared resources may exhibit economies of scale if they become relatively fewer per person, as cities grow, without impairing output. These economies of scale seem to be quite well captured by the model, using a value of  $\beta = 5/6 \simeq 0.8$  to match data, which suggests that the result makes sense both for  $N$  interpreted as an ensemble average and as a growth parameter over time.

- Finally, there are superlinear outputs, whose behaviour is more subtle. A production output  $Y$  may be a fraction, not of  $N$ , but of  $N^2$  because the maximum output can depend on the interactions between agents, and the agents working alone. Output may or may not depend on the volume of the city, i.e. perhaps only the number of agents or currents in the pudding, and perhaps the space they occupy in the course of their interactions; this depends on the kinematics of the detailed processes.

For every scaling benefit, there may also be scaling costs, with the opposite sign: contention for shared resources, spiralling costs of equilibration, etc. Some general comments about the mean field approximation:

- People have different jobs, capabilities, and habits. In the mean field representation, such detail is not represented explicitly, but this does not imply that they don't exist. Indeed, they must be present to account for different flavours of output. However, this is not a one-to-one association. Formally, the outputs are fractions of the total amount of possible produced work, averaged over all specializations, mechanisms, and functions. To understand why this is plausible, we need to understand that outputs are the result of combinations of jobs, in collaboration (see section 3.5). They are assumed equally likely on average across cities of the ensemble. The diversity of a city may increase with size<sup>8</sup>, but this does not matter as long as we assume that all jobs are the same. The data suggest that this does matter (see section 3.7).
- A city must have 'potential barriers', or entry costs for construction and community building, that depend on the level of technology available. This is also not represented in the model. There might be significant debt associated with building, which is invisible in this picture. These are transient responses, invisible in steady state behaviour.
- The technology at the time of building ought to affect the density of the city too: as transport improves and gets cheaper, it enables greater distances to be covered in the same amount of time, i.e. lower density. Conversely, it enables more efficient transport of provisions to sustain a high population density. One might not expect old cities like Rome or London to be immediately comparable with Brasilia or Shenzhen<sup>9</sup>. Similar questions could apply to the pace of life. Does this depend only on the size of the city, or also on cultural norms? What if we compared Dakar with Seoul, for example? The scaling depends only on relative density however.
- To be comparable in an ensemble average, one could expect that cities would need to be comparable in productive technology, composition and wastage. A machine could do many times the work of a manual laborer, for instance, so it would not be fair to compare a city with mechanization to a city of horses and carts. The data in [2] were taken from mainly American cities, which have a relatively uniform level of technological infrastructure, and arise from similar epochs. There is no particular reason why they would be comparable to the slums of Mumbai. But this remains to be discovered.
- By distinguishing between  $N$  and  $N_I$ , it is possible to see how about the productive capacity of the city could be altered by the presence of unproductive regions. From the expansion of (18), it would seem that the higher the proportion of the population that cannot contribute to a process, the lower the scaling power of outputs might become.

It is known, for instance, that a typical '80/20 rule' (power law behaviour) is almost universal for networks, i.e. 80 percent of the yield typically comes from 20 percent of the agents [11, 18],

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<sup>8</sup>This could be checked by comparing the size of yellow pages directory for a community to its white pages directory.

<sup>9</sup>Just 30 years old, 'Shenzhen speed' is the stuff of legend in China.

and the effect of certain hubs on transmission is crucial to understanding effective partitioning of regions [19]. This suggests that a graph theoretic explanation can add to the simple mean field picture.

Coarse graining extracts only the most generic features of a system, in the limit of large  $N$ , i.e. at the coarsest scales. It is unable to probe the separation of such scales, by weak local interactions, or distinguish long and short range interactions. At smaller range, the semantics of interactions often become vital to the functioning of a system. In physics, ‘semantics’ are simple and mostly fixed by ‘physical law’, e.g. they manifest as different ‘charges’, ‘forces’ or allowed interaction types. In a human-technology system, however, there are many more flavours of interactive behaviour that may be distinguished, and their number and patterns might even change over time.

The urban scaling predictions reviewed here have been partially matched to data, with  $N$  on the order of  $10^3$  to  $10^7$  [1]; this offers strong evidence that underlying semantics of comparable cities are unimportant at these scales. However, the argument above underestimates some of the data significantly, especially where the data exhibit a high level of uncertainty, estimated in the margins for  $\beta$ . This suggests that there is something not captured adequately by the preceding argument.

There are two possibilities: either inhomogeneities across the ensembles are significant, or the outputs do not all follow a single model, and we are looking for either an addition or a refinement. I shall try to shed some light on these issues, in the next section, by deriving the model from a more microscopic model, using promise theory.

## 2.9 Does innovation characterize superlinearity?

Some aspects of the data, analyzed in [2], are suggestive of a fit with the single universality model in [1], but not all. The authors attributed the superlinear scaling quantities to creativity or innovation [1, 2], though the link between this explanation and the calculation of the exponent is incomplete. Below are approximate representations of few samples of the superlinear scaling data from [2] (see reference for details of the numbers):

MEASURE	APPROX. AVERAGE $\beta$	SOURCE
Wages	$1.12 \pm 0.02$	USA
GDP	$1.13 - 1.26 \pm 0.1$	EU, Germany, China
Patents	$1.27 \pm 0.02$	USA
Private R&D employment	$1.34 \pm 0.05$	USA
R&D employment	$1.26 \pm 0.1$	china
R&D establishments	$1.19 \pm 0.03$	USA
AIDS cases	$1.23 \pm 0.05$	USA

As pointed out by the authors, many of the numbers have a high level of uncertainty, due to the difficulty of fitting data from disparate sources<sup>10</sup>. Even with generous margins, the single predicted value of  $\beta = 7/6 = 1.16$  does not plausibly agree with all of these measures, however, and the deviation seems to not be attributable to a normal variation. The data in the table below, by contrast, were collected in an independent study in the UK that attempted to verify the superlinearity hypothesis in only a single measure: patents (see reference [13] for details):

MEASURE	APPROX. AVERAGE $\beta$	SOURCE
Patents	$1.13 \pm 0.1$	UK all cities
Patents	$0.99 \pm 0.1$	UK small cities

<sup>10</sup>My hat goes off to the authors for making this effort though!

The authors of this study point out the difficulty in defining what constitutes a valid city for the purpose of ensemble comparison. Those values are significantly lower than those in the first (SFI) table. While the SFI data for patents do not agree with Bettencourt’s model, the first set of UK data agree much better (1.13 is close to 1.16), but the second restricted set do not (0.99 instead of 1.16). These discrepancies warrant an explanation.

From the perspective of promise theory, which I will apply in more detail in the next section, one thing stands out about the measures in the first table: the outputs are of two qualitatively different types: some measures are ‘promises’ and some are ‘agents’.

- Patents, wages, GDP, disease, are related to the promises of an output, i.e. produce that relate to a production process.
- Jobs related to research are not outputs but occupations. They represent the state of agents. They are not (directly) the result of a process<sup>11</sup>.

If we consider the version model of [1], there are two freedoms that remain to alter (17). One is to imagine the existence of a large fractional  $N_0$  population (e.g. by invoking the 80/20 rule for the distribution of outputs), relative to the population  $N_I$ , involved in making each particular promise. However, this would have the effect of reducing the value, so, while it might potentially explain the UK data, it could not explain the SFI data. The second is to assume  $H > 1$ , which suggests a significant self-similarity in the infrastructure that relates to patent production. However, this does not seem credible as the infrastructure that enables patents is principally researchers, which do not think in paths that fill space. Something about the universality assumptions need to be reconsidered. A possible resolution is provided in section 3.7.

One clue to these behaviours could be that the selections represent highly specialized sub-populations, rather than homogeneous fractions. Recall that the way superlinearity arises is that an efficiency of scale effectively gets better at large  $N$ , leading to an effective amplification with the size of the population. Wages and GDP to involve the largest fraction of the population of a city community. All the other measures are based on highly specialized populations to which few contribute. Another point is that patents are not merely the result of an economic process, but are sometimes weapons used politically and strategically as part of non-collaborative economic warfare between companies. This suggests that semantics would play a role in their explanation, and potentially skew some data (cities with companies like Apple and Samsung, well known for software patent fights<sup>12</sup> might appear differently).

In section 3.7, a generic possible explanation is proposed, based on the generalized semantics of scaling agents into clusters (superagents). If one distinguishes agents from their promises, then staged efficiencies, arising from functional dependency, can be compounded or reduced, altering the  $\beta$  value hierarchically.

$$\text{Infrastructure} \xrightarrow{\times 1.16} \text{Superagents} \xrightarrow{\times 1.16} \text{Produce/Output} \quad (20)$$

To derive this plausibly, we need to formulate a simple promise theory of communities.

## 2.10 Gunther’s ‘Universal Scaling Law’ and spacetime involvement in scaling

Before looking at a deeper model, I want to comment on another scaling model, from the world of information technology (IT). Superlinear scaling has also been observed in high performance cluster computing [20], albeit for a different reason. Most IT models are essentially one dimensional in nature, describing serial processes, divided into parallel one-dimensional threads. The amplification of output in this kind of flow network is based on queueing results, which may approximated quite well by a simple

<sup>11</sup>They could conceivably be related to a training process, but then should be counted as population or output?

<sup>12</sup>Software patents usually have low production costs, as they are often trivial and frivolous inventions.

current flow models, when workloads patterns are sparse and evenly insensitive to mixing (e.g. Poisson, or Lévy processes) [21].

Amdahl's law, and Gunther's generalized 'universal' scaling law [20, 22, 23], describe the 'speedup', or amplification of output, in a basically serial process, by the addition of workers. Jobs may be speeded up, but are ultimately limited by the serial coordination of work. Suppose we have  $N_L$  local, parallel worker threads; then this takes the general parametric form:

$$S(N_L) = \frac{N_L}{1 + \alpha(N_L - 1) + \beta N_L(N_L - 1)} \quad (21)$$

The output rate for a collective of  $N_L$  agents is  $S(N_L)$  times that of a single agent (see appendix D). The two terms in the denominator represent two kinds of network process, at close range. The linear term is a bottleneck term, where all the agents contend over a limited shared resource. As long as  $\alpha > 0$ , scaling will be sublinear. The quadratic term is the cost of making the mesh of agents agree about something at each stage of the process. This is a cooperation, or 'homogenization of information' (coherency) term, which puts a sharp brake on scaling. For example, in database or cache replication, information has to be replicated consistently, which is expensive. This term reduces the speed up to the point where performance can actually grow worse with increasing  $N$  (see figure 4).

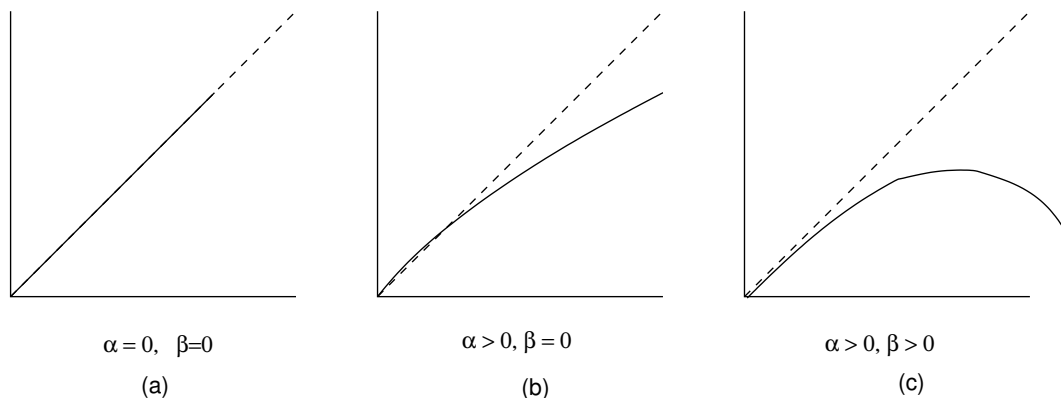


Figure 4: The Gunther universal scaling law for its control parameters. (a) linear scaling, (b) cost of sharing resources and diminishing returns from contention, (c) Negative returns from incoherency and the cost of equilibration.

The expression (21) cannot exhibit a superlinear result for positive coefficients. Nonetheless, superlinear scaling has been observed in computing clusters [20]. How can these facts be resolved? Gunther argues that superlinear scaling is not possible without violating work conservation; but, by artificially making  $\alpha < 0$ , as a parametric fit, it is possible to simulate higher dimensional effects in this one dimensional projection. This is a one dimensional view of a process that is actually two or three dimensional. Datacentre clusters, with dense interconnect networks approach mesh-levels that fill the effective volume of the datacentre during their internal processes<sup>13</sup>. As this trend continues, we can expect datacentres to behave a lot like the model of cities discussed.

The appearance of superlinear scaling came as a shock in IT, because there is no way to understand  $\alpha < 0$  from a microscopic model. However, it can be understood as a renormalization effect, i.e. an effective parametric representation of the projected output. Qualitatively, the serial limit on scaling could

<sup>13</sup>This is called East-West traffic in the IT industry.

be cheated as follows, in a sparsely utilized system. When the average state of the system is highly underutilized to begin with, then some of the efficiency can be regained by close packing of the utilization<sup>14</sup>. If dense packing can absorb the increased size of a system (like a city or computational process), then that economy of scale can enable greater output for a relatively smaller infrastructure (renormalized negative growth). Could this lead to a superlinear speedup? The phenomenon of packing is related to queue parallelization, where for example  $G/G/N$  is provably more efficient than  $G/G/1 \times N$  [4, 24]. This is because wasted idle time can be eliminated by arranging by an efficient packing of work. This cannot persist for ever, as eventually the sparse utilization of the limiting work agents must fill up to capacity. As it does so, the amount of contention ( $\alpha$  for shared bottlenecks, and  $\beta$  for equilibration of state [25]) must increase rapidly. The response time  $R$  of a queue (proportional to its average length divided by the dispatch rate  $\mu$  takes the form:

$$R \simeq \frac{E(n)}{\mu} = \frac{1}{\mu - \lambda} \quad (22)$$

This queue is only stable when  $\lambda \ll \mu$ , indicating sparse usage. The scaling indicated in the city results indicates cities that have not peaked in their infrastructure utilization yet. Superlinear scaling sounds like a good thing but it is unstable, as the queueing projection illustrates.

The rational queueing expression, in Gunther's Universal Scaling Law (21), can never explain fractional scaling exponents seen in cities, but it can demonstrate some projected superlinearity with  $\alpha < 0$ . To feed superlinearity, we need something more than parallel serial processes where the work is done by  $N$  point sources. Only if the work is done by  $N^2$  interactions can a partial efficiency make the exponent greater than unity. To get this, we need to feed higher dimensional volumes into lower dimensional volumes. This is what is going on in the mean field theory of [1].

$$\text{Interaction output} = \text{Maximum output}(N^2) \times \text{Fraction of infrastructure involved}(N) \quad (23)$$

From a graph theoretical perspective, this is a change in average connectivity of the infrastructure network (i.e. the average degree of nodes  $k$  [4]). If the fraction is a fraction of a volume rather than a line or a number, there are dimensional exponents involved, which represent the contact efficiency by close packing the city population. With more dimensions, a larger surface area can be used for interaction. If we assume that the infrastructure network is sufficiently dense that it reaches almost everyone, then this continuum approximation is reasonable.

$$\text{density of infrastructure users} = \frac{N_I}{V_I} = n_I N^{\delta(D)}. \quad (24)$$

where  $\delta(D) = 1/D(D+1)$ , for  $H = 1$ . Recalling that this volume is really a continuum approximation of a network of nodes, this translates into an average node degree utilization (or locally used connectivity) within the infrastructure channel<sup>15</sup>

$$k(N) = \frac{N_I}{V_I} = n_I N^{\delta(D)}. \quad (25)$$

Assuming the infrastructure is pervasive so  $N \simeq N_I$ , the equivalent serialized infrastructure volume, for a single process, is something like:

$$\begin{aligned} V_I &= \left(\frac{V}{N}\right)^\delta \times N_I \simeq N^{1-\delta} \times \text{cross section}, \\ &= \text{capture volume per agent} \times \text{span of agents} \times \text{cross section}, \end{aligned} \quad (26)$$

<sup>14</sup>This is the motivation for packetization (atomization) of networks, and context switching in time sharing operating systems.

<sup>15</sup>Note that this is not a real connectivity, which has to do with the number of nodes, but a kind of close-packing of the sparse interactions that occur between the nodes into the infrastructure stream.

Using this volume, instead of the total volume of the system (city, community, etc), recognizes two things. First that cost of the infrastructure is much less than that of the entire system; and, second, that it is the serialization of the sparse resource over a standard cross section that we want to use for comparable work output. This is like fitting the sparse output volume of the city into an idealized serial stream of fixed width to see how its length scales with the number of inhabitants<sup>16</sup>. The efficiency comes from being able to use more of an unexpected fixed cost, sparsely utilized resource along with other economies of scale. The net result is an amplification of the output by  $\delta$ :

$$\text{Interaction related output } Y = \frac{\text{const}}{V_I} N_I^2 \simeq Y_0 N^{1+\delta(D)}. \quad (27)$$

The numerator is unexpectedly constant, but the infrastructure volume scales sub-linearly, the net output appears superlinear, with these assumptions. The question is how do we know if these are the same assumptions as the used for the measurements?

Modern datacentres and networks at scale have multiple redundant paths that make their interconnection networks space filling (e.g. Clos structures [26]). This brings higher dimensional scaling issues into the picture. The general principle is one of close packing of utilization. When dependencies scale more favorably than the contended processes that rely on them (relatively speaking), each process gets a larger share of the shared resource, and is accelerated for a while, provided the total utilization remains low.

### 3 A promise theory derivation of the city model, and beyond

I now want to show that we can re-derive the scaling, and indeed the model of [1], from using a quasi-atomic theory of a network, taking into account some of the more important semantics to see how universality emerges, and where its limits might lie. Promise theory strikes a balance between semantics and dynamics, and thus between coarse graining and the chemistry of different functional agents.

#### 3.1 Brief recap' of promise theory

Promise theory is a formalism that describes dynamics of atomic, black-box agents, alongside their broad functional semantics, i.e. with their intentional behaviour. The default or ground state of agents is one in which they make no promises, and are independent or 'autonomous', i.e. each agent is self-sufficient and controls its own internal resources, and each agent can make promises only about its own capabilities and intentions<sup>17</sup>. Assuming that an agent is not deliberately deceiving, a promise may be considered the best available local information about the likelihood of an outcome.

Promise theory can be understood in a number of ways. For the present purpose, it can be thought of as a labelled graph theory, with some rules and constraints on interpretation. A promise from an agent  $S$  to one of more agents  $R$  takes the form

$$S \xrightarrow{b} R \quad (28)$$

where  $b$  is the body of the promise. The promise body, explains what is being promised, and has polarity (+ to offer, and - to accept), as well as type  $\tau(b)$  and constraint  $\chi(b)$ . All agents are considered to be

<sup>16</sup>A simple analogy is to think of a tube of toothpaste. The toothpaste comes out in a one dimensional stream of fixed width, but we are forcing the output of a three dimensional tube through this portal, and asking: how does the amount that comes out increase with the size of the tube if we squeeze it in the same way? By fixing the cross section, we can compare different tubes, or different cities.

<sup>17</sup>From a physics perspective, promises look a lot like a rich array of charge flavours, for exotic forces, and the network looks a lot like a force field.



indistinguishable to begin with, and these agents may work together to form aggregate ‘superagents’, by promising cooperation [9].

In a community or city, we might imagine that the atomic agents are persons, machines, and even software, acting as proxies for human intentions. An aggregate ‘superagent’, such as a company can collaborate to keep new promises (see figure 5). Each aggregation into a superagent enables new promises to exist that cannot be attributed to any of its components. To understand output yields, the key is to ask: what kind of agent is responsible for an output? Is it a single point source agent, or a distributed superagent, with an internal collaboration mesh? In either case, collaboration has a finite range. In

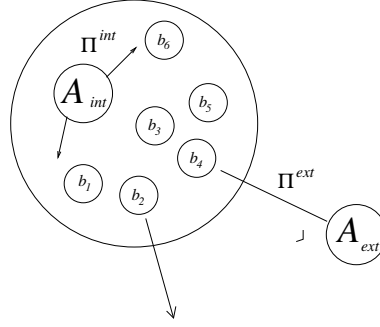


Figure 5: What is an agent? Agents aggregate to make superagents, with new promises possible at each scale of agency. So what we consider to be an agent, depends on the scale at the networks under consideration.

summary:

1. Agent types are distinguished by the promises they make.
2. The way that agents make use of one another's intent through promises is what we mean by agent semantics.
3. The outcome of a promise is deduced by observation; this is called an assessment in promise theory. An assessment by agent  $i$  about a promise  $\pi$  is written  $\alpha_i(\pi)$ .
4. A link between two agents requires a promise by both parties: one to make a service offer (+), and the other to accept it (-).

$$\left. \begin{array}{l} S \xrightarrow{+b} R \\ R \xrightarrow{-b} S \end{array} \right\} = \text{unidirectional transfer} \quad (29)$$

This kind of binding is the basis for determining the coupling strength  $g_Y$  for the output yields.

Promise theory is a theory of incomplete information, and embodies controlled coarse graining over semantic scales.

### 3.2 Agents and super-agents, long and short range interactions

Consider a set of agents  $A_i$ , where  $i$  runs over the population of the system (city, datacentre, etc), and all machines and proxies for human intent. In order to collaborate, agents need to make some basic promises [9]. Every promise may be either kept or not kept, and the average value needs to be self sustaining. Each autonomous agent thus has a balance of payments to consider. It needs to accept fuel,

food, energy, money, etc. Dependencies also include raw materials which have to come from outside. If agents can cache resources they can maintain weak coupling, else they are strongly dependent on their environments. This applies to energy, supplies, and also inputs of information and ideas. Promises made directly between agents are called short range.

**Definition 1 (Short range interaction)** *A binding between adjacent agents  $S$  and  $R$  of the form*

$$S \xrightarrow{+\tau, \chi_+} R \quad (30)$$

$$R \xrightarrow{-\tau, \chi_-} S \quad (31)$$

where  $\tau$  is the same in both promises, and  $\chi_+ \cap \chi_- \neq 0$ .

A promise may be long range if it is non-local, i.e. it couples several agents together, or employs intermediaries.

**Definition 2 (Long range interaction)** *A binding between adjacent agents  $S$  and  $R$ , through an intermediate agent  $A$*

$$S \xrightarrow{+\tau, \chi_+ | d} R \quad (32)$$

$$R \xrightarrow{-\tau, \chi_-} S \quad (33)$$

$$I \xrightarrow{+d} S \quad (34)$$

$$S \xrightarrow{-d} I \quad (35)$$

where  $\tau$  is the same in both promises, and  $\chi_+ \cap \chi_- \neq 0$ .

Note that there is no a priori notion of distance in a graph, other than the number of interactions or hops between agents nodes. The familiar notion of distance comes about from embedding a graph in metric space, which in turn is related to a continuum approximation.

At any scale, a promising agency that plays a functional systemic role makes promises of the following general forms.

$$A_i \xrightarrow{-f} A_{\text{ext}}, \quad \forall i. \quad (36)$$

There must be an external agency acting as a source of this fuel  $f$ , providing

$$A_{\text{ext}} \xrightarrow{+f} A_i, \quad \forall i. \quad (37)$$

Any agent  $A_i$  may depend on pre-requisite promise of dependency  $D$ , provided by another, in order to provide service  $+S$ ; according to the assisted promise law [9]:

$$A_i \xrightarrow{+S | d} A_j, A_i \xrightarrow{-d} A_k \simeq A_i \xrightarrow{+S} A_j \quad (38)$$

provided

$$A_k \xrightarrow{+d} A_i. \quad (39)$$

In this way, agents have probes, services, skills (+), and needs or receptors (-) that can unlock or catalyze their functionality. The expresses its exterior promises outwardly, e.g. a door handle's function is recognized by its shape, just as a car and its promises are recognizable by its exterior structure. Interior promises might be involved in making the exterior ones, but these are not generally visible at super-agent scale. The basics of scaling semantics and agency are laid out in [7].

### 3.3 Promise networks: functional interactions

Functional networks have two aspects to their productivity, which can be described by:

- Replication of agent output: dynamics, economics.
- Combination of used services, ideation by mixing: intent, semantics, fitness for purpose.

It is the combination of these that leads to the understanding of fully functional scaling. Basic communication and supply infrastructure networks enable interactions between any pair of agencies, but specialist functional networks are typically small and disconnected [27]. They relate specific promises or services that are constrained by operational semantics. The scaling of such networks has been examined in [7]. It has a few aspects:

- Agents keep promises at scale by individually promising similar capabilities in parallel, e.g. Amdahl's and Gunther's laws of scaling [20, 22, 23]. A promise that is considered kept by a single agency scales by having multiple sub-agents form a 'superagent'. Now Bettencourt's insight reveals how output can also scale in a non-parallel fashion [1].
- When one agent depends on a promise being kept by another, in order to keep a promise, this creates a serial dependency, introducing a queue and a handling rate scale. The agents must be able to understand one another, with common language, else they are partitioned. Partitioning cuts off long range interaction, and promotes long range diversity, like cultural and specialist diversity<sup>18</sup>.

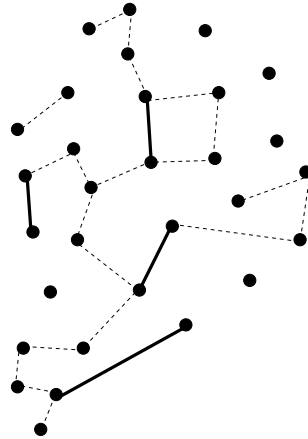


Figure 6: No single type of promise binding (dark lines) leads to percolation of value in the promise graph. However, with conditional dependency, and sufficient diversity and homogeneity, there can still be effectively close to  $N^2$  links whose value converts into a common currency.

### 3.4 Forces that condense a cluster

A hypothesis of promise theory is that one may define a notion of force for agents, which is attractive when there is economic advantage, and repulsive for economic disadvantage<sup>19</sup>. The formation of super-

<sup>18</sup>The lack of a common language is effectively a channel separation, disconnecting networks into separate branches. In communication networks, channel width is sometimes shared between different partitioned agencies by using non-overlapping frequency ranges. This is called Frequency/Wavelength Division Multiplexing (FDM). It is a form of multi-tenancy.

<sup>19</sup>It does not matter here whether we consider the force to be a Newtonian deterministic force, or a probabilistic susceptibility for drifting closer, as in stochastic systems. We can think of a generalized energy-momentum tensor [28], for generalized agent 'fields'.

agents thus comes about, for economic reasons [29], by the value of collaboration. If the promises are unconditional, superagents will be localized. If they are conditional, the clusters are ordered and may thus be distended or even scattered.

- Agents, which make the same kinds of promises of same polarity, tend to repel one another.
- Agents, which make complementary (binding) promises of opposite polarity tend to attract one another.

Applied to the city problem, this suggests that the basic attraction to condense the city out of a surrounding gas of agents comes from the common supply promises, which are predominantly of positive sign, and that all agents share; for the survival infrastructure. This is held in check by the repulsion of agents making similar promises, except where there are promises to cooperate. One may expect structures as follows:

- Clusters of professionals bound together by cooperation promises.
- Chained transport agents, bound together by conditional promises.
- Distributed competitors, perhaps clustering around shared infrastructure hubs, e.g. malls, districts.

The embedded spacetime structure of the city should be an equilibrium configuration between the attraction to needs and desires, and the repulsion from competition with similar skills.

In promise theory, a specialized role characterizes a pattern of agents that make similar promises. By specializing specific tasks to specific agents, each agent can be more focused in learning and adapting, but acquires an additional cost of cooperation proportional to some positive power of the cluster size. Superagency collaboration is a *short range* interaction (between interior promises) [7]. It can form *long range* strong interactions, with associated cost, if it has the internal resources to support these. Limiting interactions to short range leads to stability. The long range interactions drive superlinear effects, but also promote ‘chaos’.

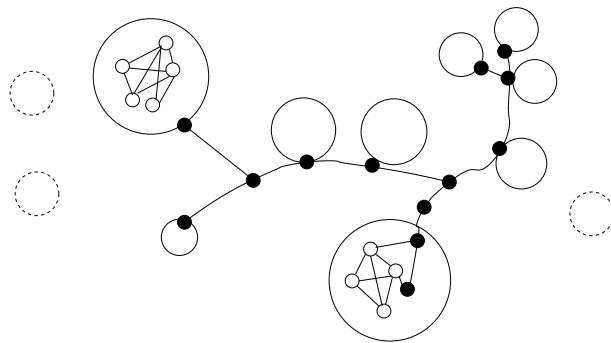


Figure 7: The geometry of superagents may fill space in different ways. Infrastructure that interconnects other agencies is a superagent in its own right, involving linear or approximately linear cooperation between member agents. Under preferential attachment, agents  $N_I$  tend to cluster around the infrastructure agency, leaving a few  $N_0$  padding out the spatial volume. The circles around the subagents may be considered infrastructure binding the agents together.

The notions of attraction and repulsion are wired into our imaginations in terms of spatial concepts. Even without an embedding spacetime, we can speak of agent affinities, like the interactions described in molecular chemistry, where spacetime plays no real role. With a physical volume to embed a graph of

promise-keeping interactions, geometry ties range to distance, but in a virtual network (which includes transport of messages by intermediate carrier), short range interactions can also be disseminated over a longer effective range, by adding cost or latency (such as in telecommunications).

### 3.5 Promise networks that percolate

In order to justify Metcalfe's law, there needs to be an average level of interaction that spans the complete graph and propagates value. This doesn't necessarily require a single promise type to dominate the entire graph, because it is the value graph, not the promise graph that needs to link up all the nodes. However, from the previous section, we would expect basic infrastructure to dominate. The main requirement for this is the presence of a common (or at least interchangeable) currency between all the nodes.

Specialized exterior service promises naturally lead to small molecular clusters of component 'atoms' (superagents), that make specific interior promises. They seldom span large areas, because promises act like short range interactions (which is also why superagents can be considered quasi-atomic black box agents, see figure 6), so they do not easily bring about percolation of value in an economic zone, like a city. The promises, which are ubiquitous, are associated with the survival of agents, and relies on the most general kind of infrastructure in systems: power, food, air, water, etc. These are likely responsible for the interconnection of the many smaller microcosms of value creation (small businesses in cities, and microservices in IT) to bring about a unified community with its economies of scale.

If we let  $N_\tau$  be the number of agents that consume a promise of type  $\tau$ , then we expect the class of  $\tau$  related to survival to be of the order  $N_{\text{survival}} \simeq N_I$ , in the meaning of the city model. For all other types,  $N_{\text{other}} \ll N_I$ . However, we'll see in section 3.7 that long range interactions are also needed to explain the scaling exponents for cities. The size of the effective network is not therefore given by the adjacency matrix of the underlying infrastructure network, but rather by the typed promise graph.

It is useful to recall the definition of a promise network (see [7]).

**Definition 3 (Promise adjacency matrix)** *The directed graph adjacency matrix which records a link if there is a promise of any type  $\tau$ , and body  $b_{ij}(\tau)$  between the labelled agents.*

$$\Pi_{ij}^{(\tau)} = \begin{cases} 1 & \text{iff } A_i \xrightarrow{b_{ij}(\tau)} A_j, \\ 0 & \end{cases} \quad \forall b_{ij}(\tau) \neq \emptyset \quad (40)$$

The adjacency is the effective topology of the spacetime network, as far as the agents are concerned. The link-occupancy of this matrix, for a given promise type, is a linear sum whose value is generally much lower than that of the total possible mesh of interactions. Thus, for any promise type  $\tau$ ,

$$\sum_{i,j=1}^{N_I} \Pi_{ij}^{(\tau)} = N_\tau(N_\tau - 1) \ll N_I^2, \quad (41)$$

Note that an agent can make a promise to itself too, so the upper limit could be written  $N_I^2$ . The value-percolating connectivity or degree of a node

$$\Pi_{ij} = \sum_{\tau} \Pi_{ij}^{(\tau)}, \quad (42)$$

$$k_i \simeq \sum_j \Pi_{ij}. \quad (43)$$

We can also write this in terms of the direct valuation of the promises, in terms of the actual matrix of promises  $\pi_{ij}$  [7]:

$$k_i \simeq \sum_j v_C(\pi_{ij}). \quad (44)$$

where  $v_C$  is the value of the promise as calibrated and assessed by a common central agency (see appendix A).

Agents can keep multiple promises and multiple types of promise ‘simultaneously’ over a given timescale  $T$ , by multiplexing their time at a rate that is much faster, i.e.  $t_{\text{multi}} \gg 1/T$  to avoid the queueing instability. On the assumption of sufficiently sparse packing:

$$\sum_{\tau} \sum_{i,j=1}^N \Pi_{ij}^{(\tau)} \leq N(N-1). \quad (45)$$

For the economic output of a promise network, we care more about the assessments of which promises were kept than the number of promises that were made (see appendix A). Each agent assesses promises individually, and they may not agree. However, to compare to city statistics, we may assume that an statistical bureau agency has been appointed by the city to calibrate these assessments  $\alpha_{\text{official}}(\pi_{\tau})$  according to a single scale. Promise-keeping is an average over time. Provided the sum time to keep a promise, for all  $\tau$ , for each agent, is much less than each time interval of the assessments, we can reduce  $\alpha(\pi)$  to a frequency ‘probability’. Another way of saying this is: provided the cost of keeping the promises is less than the budget of each agent.

These estimates are maximal. The size of a functional cluster is not really related to any of these graphs, because there are semantic constraints. Specific functional behaviour, in a single promise type, is a strict limitation, which leads to very sparse subgraphs. To gauge an average measure of the total economic impact of all functional interactions, we have to assume:

- The functions are successful in driving an economy.
- The density of implicit interactions is quite high, else a given output  $Y$  will not be represented by an average mesh density.
- There are some long range interactions that make the partially connected graph totally connected on average, even if only at a low level. The survival promises probably fulfill this role.

In reality, a city or community might be partitioned into quite independent regions, leading to a modular reducible form [30]. If one imagines the specific network, which delivers output  $Y$ , it may be some maximally quadratic polynomial of  $N_I$ , related to the structure function of the network, but it may also be significantly less than this. What will tend to lead to percolation of interactions, which bind in a mesh, is the existence of long range, pre-conditional promises, i.e. dependencies. Then, the sum will be some polynomial of  $N_I$ , such that

$$\sum_{\tau} \sum_{i,j=1}^N \Pi_{ij}^{(Y)} \simeq c_1 N + c_2 N^2. \quad (46)$$

If  $i, j$  run over all the individual agents within city limits, then these matrices are sparse and fragmented for each  $\tau$ . Suppose we assume that there is no dominant infrastructure, only small clusters of voluntary cooperation, as in [27]. Then, the aggregate graph for a city output  $Y$ , would need to have sufficient random cooperative connectivity to form a process that generates output algorithmically. The density can be estimated, if we assume that, for promise type  $\tau$ , an agent has an average valency  $\mathcal{V}$ , then we would need

$$\sum_{\tau=1}^{\tau_{\max}} \sum_{i,j}^{\mathcal{V}_{\tau}} \alpha_C \left( \Pi_{ij}^{(\tau)} \right) \simeq N \mathcal{V} \tau_{\max} \leq N^2. \quad (47)$$

Moreover, since the diversity of promise types is unlikely to be greater than the population it stems from, one estimates  $\frac{N}{V} \leq \tau_{\max} \leq N$ , which does not seem realistic. Taking these estimates into account, it seems most likely that a few types of promise relationships dominate the connectivity and value creation, i.e. the basic ‘survival’ (food, water, sanitation, communications) promises, and the specialized outputs contribute relatively little to the total output, except when directly dependent on the survival promises. This is a further suggestion that the attraction to urban life stems from core interconnection infrastructure, rather than from independent diversity.

### 3.6 Scaling of roles and interactions

Consider now the role of dependence in keeping promises as a determinant for scaling.

#### 3.6.1 Linear consumption by independent agents

For promises that are requirements for survival, every agent more or less deterministically depends on some source infrastructure agent  $S$ , the number of promises is one to one:

$$\text{Number of consumed} = \sum_{i=1}^{N_I-1} \Pi_{iS}^{(\tau)} \simeq N_I - 1. \quad (48)$$

If the source is distributed over several agents, then this is still true:

$$\text{Number of consumed} = \sum_{s=1}^S \sum_{i=1}^{N_I-1} \Pi_{is}^{(\tau)} \simeq N_I - 1. \quad (49)$$

Thus, the number of promises needed to supply this demand is also proportional to  $N_I$ :

$$N_{\text{infra}} \equiv N_{\tau} \simeq N_I. \quad (50)$$

#### 3.6.2 Sublinear economy of scale, and spacetime involvement

When agents interact through links, on a small scale, the chemistry of their interactions may be based on simple counting. Normally, in promise theory, we count by agent or by link. However, as the numbers of converging links become so great that counting is impractical, there is no way to liken a process to a simple Poisson arrival queue, and we resort to flow counting based on density arguments. Let’s now show that this is equivalent to volume of the infrastructure  $V_I$  in [1]:

Consider a number of agents  $N_{\text{infra}}$  who provide infrastructure (gas stations, supermarket, etc) for a number of clients  $N_{\text{client}}$ , which in turn offer a service conditionally, based on the dependence.

$$\pi_{\text{infra}} : A_{\text{infra}} \xrightarrow{+\text{infra}\#\mathcal{V}} \{A_{\text{client}}\} \quad (51)$$

$$\{A_{\text{client}}\} \xrightarrow{-\text{infra}} A_{\text{infra}} \quad (52)$$

$$\{A_{\text{client}}\} \xrightarrow{+\text{service}|\text{infra}} A_{\text{infra}}. \quad (53)$$

Suppose that each infrastructure agent  $A_{\text{infra}}$  can promise to service  $\mathcal{V}$  clients simultaneously; then, using a simple valency argument, we have a detailed balance equation for the interactions at steady state:

$$\alpha_+ N_{\text{infra}} \mathcal{V} \geq \alpha_- N_{\text{client}}. \quad (54)$$

Thus for simple counting of distinguishable agents, we may estimate the number of infrastructure agents needed to support a number of clients:

$$N_{\text{infra}} \geq \underbrace{\left( \frac{\alpha_-}{\alpha_+} \right)}_{\text{intrinsic}} \frac{1}{\mathcal{V}} \times N_{\text{client}}. \quad (55)$$

where  $\alpha_-/\alpha_+$  may be interpreted as the affinity for the service, or the reciprocal compressibility. This scales linearly with the number of clients in the catchment area of the infrastructure. Moreover, there is no way, in this detailed formulation that we can count otherwise. The only economy of scale in this arrangement is the standard linear multiplexing result for the marshalling of  $\mathcal{V}$  queues into a single queue with  $\mathcal{V}$  servers, noted in section 2.10.

However, if we now ask how to count the number of clients that can be fed into a single infrastructure agent, in a spatial volume, with dimensional multiplexing, then the best estimate is to serialize the counting, as before:

$$N_{\text{client}} = \left( \frac{V_{\text{catchment}}}{N_{\text{users}}} \right)^{\frac{1}{D}} C(D) \times N_{\text{users}}, \quad (56)$$

where we imagine a catchment volume  $V_{\text{catchment}}$ , containing any number of agents  $N_{\text{users}}$  who are interested in the infrastructure service, and we serialize them along a tube of constant cross section  $C(D)$  (see figure 8). Although these numbers only apply to a small mesoscopic volume of space, in a homogenous

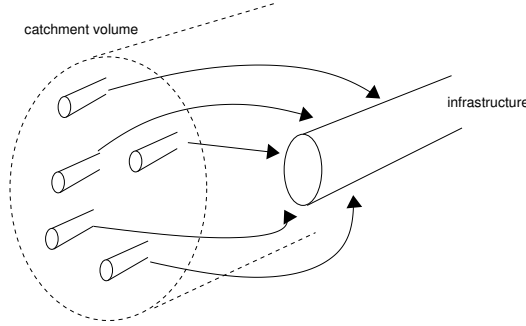


Figure 8: How spacetime involvement compresses serialized agent links into an effective flow of fixed cross section.

city, this will apply to the entire city, so we are justified in taking

$$N_{\text{use}} \simeq N_I, \quad (57)$$

which, combined with an equation of state for the volume, reproduces the earlier result

$$N_{\text{infra}} \simeq V^{1/D} N^{1-1/D}. \quad (58)$$

In this argument, it is clear that only the active agents  $N_I$  play a role in the counting, and process flow, hence this also justifies why we can assume  $N_I \rightarrow N$  in [1].

### 3.6.3 Deriving Metcalfe's law from promise networks: the importance of conditional dependency

A key assumption in the scaling argument of city outputs in [1], is what is Metcalfe's law [14–16], which proposes that the value of a network scales like the square of the total number of nodes, i.e. value is



generated by the number of possible links between agents<sup>20</sup>. This has received some empirical support in [17], but has also been criticized in [15, 16]. More importantly, interaction value generation is not the same as output. Agents can also produce wealth without interacting, if they have all the prerequisites (short range interactions). In section 3.7, we'll see that long range interaction forms the basis for one kind of value creation, but not the only one.

Promise theory predicts that links represent value in the following way. Consider then the sum of all impartial promise valuations by third party  $C$ . If we assume that all agents assess the value of interactions as strictly positive, then:

$$\text{Mean value} = \sum_{\tau} \sum_{i,j=1}^{N_I} v_C(\pi_{ij}^{\tau}) \leq c \langle \alpha_i \alpha_j \rangle N_I (N_I - 1) \quad (59)$$

where  $N_I = \max_{\tau}(\dim(\tau))$  (see appendix A about promise valuations). In spite of the quadratic appearance from the result, this is a linear sum, so it acts automatically as a linear averaging measure. Also, for any given specialization  $\tau$ , the filling fraction of the promise network is likely low; thus, a key assumption is that, when properly documented, agent's specialized promises in fact depend on many others *conditionally*, forming a wide reaching network of progressively weak coupling. Conditional promises propagate the range of value interaction [9, 31]. This is the ecosystem effect.

The weakness of coupling is not a problem provided the city is reasonably homogeneous in density over the timescales of the ensemble parameters, because of the assumption of overall sparseness. If we define an effective density for the network, which describes some probable average level  $\rho \in [0, 1]$  of 'intercourse' between agents (any kind of sustained relationship), then it is fair to write the value of a network of promises:

$$\text{Mean value} = \sum_{\tau} \sum_{i,j=1}^{N_I} v_C(\pi_{ij}^{\tau}) = c \rho N_I (N_I - 1). \quad (60)$$

provided the total density of promises forms an SCC of order  $N_I$  members. This value can be distorted from the quadratic form by significant inhomogeneity. Now, for most cities,  $N_I \gg 1$  and  $\rho, \alpha_i \ll 1$ , so for strictly positive value interactions:

$$v \simeq c \rho N_I^2. \quad (61)$$

This is Metcalfe's law. It depends on the assumption of strictly positive value (i.e. no non-profitable interactions), and sufficient density of promises to involve everyone in the city who belongs to the infrastructure. Why is this plausible, when most specialization leads to modularity? One reason is that modularity is only a separation of scales, not an elimination of dependency: dependencies form an ecosystem. Nearest neighbours might hold the greatest semantic importance to a given function, but this reductionist viewpoint is not independently sustained without the eigenstability of the entire web [30].

### 3.7 Conditional dependency as an explanation for multiple superlinearity classes

We can note briefly why certain system processes (or occupations, in a city) scale differently. In a specialization society, singular individuals or agencies rarely have all the prerequisites to complete their work without assistance. They need to collaborate and depend on others. Thus other agents take on the role of effective infrastructure for one another. It is the accessibility of this dependency on one another that throttles output, and can modulate scaling behaviour.

<sup>20</sup>Metcalfe's law does not refer to the cost/value of physical connectivity, which (once again) can be much sparser than a mesh at low utilization. Indeed, that is important, else the net profit approaches zero. Rather, it refers to a correlation of agents' promised activities, linking their behaviours and generating value by interaction.

The argument for superlinear scaling in cities in [1] uses interactions, with arbitrary (unknown mean field) range that can span the city, to model economic output. However, from a promise theory perspective, it is reasonable to ask the question: should we consider the output to be a fraction of  $N^2$ , representing the maximum output due to interactions (as in Metcalf’s law), or should we consider the output to come directly from the agents, as a fraction of  $N$ , as in Amdahl’s law. This offers an explanation for the anomalous superlinear exponents in the data for [2]. The superlinear scaling was initially associated with ‘innovation’ activities<sup>21</sup>. However, the promise theory shows how one does not need to invoke a process of innovation to explain the scaling. To drive the long range cohesion of the whole community network, specialists come to depend on specialized services (e.g. patents depend on lawyers). This leads to a number of cases:

1. **Interaction scaling:** as proposed in [1], for interactive value creation.

$$A_{\text{Lab}} \xrightarrow{+\text{patent}} A_{\text{observer}} \quad (62)$$

$$A_{\text{Lab}} \xrightarrow{\pm\text{interact}} A_{\text{services}} \quad (63)$$

Patent agencies are interacting at arbitrary range with a significant fraction the total promise graph, as a part of the ecosystem.

$$Y \simeq Y_0 \left( \frac{v}{V_I} \right) N^2 \rightarrow N^{1+\delta} \simeq N^{\frac{7}{6}} = N^{1.16}. \quad (64)$$

The amplified value relies on the interplay between long range mixing, and short range isolation.

2. **Scarce agent scaling:** skilled specialist experts’ output is proportional to the number of skilled agents, since their queue is sparse, and not filled by a wide volume of demand. However, the same economy of scale applies to their services when they are depended on, as ‘infrastructure’, by others.

$$Y \simeq Y_0 \left( \frac{v}{V_I} \right) N \rightarrow N^\delta \simeq N^{\frac{1}{6}} = N^{0.16}. \quad (65)$$

3. **Interaction promises with a scarce dependency:** such as in the case of a service that depends on a source of agents to fulfill a dependency. e.g. patents can only be produced by labs that depend on the outputs of specialized R&D employees and lawyers, working in private relationships, or in secrecy. The expression in (76) assumes a promise configuration like that of the assisted promise law [9], with a main output based on a number of agents that provide input. The dependencies produce raw output, and the ‘lab’ agency collates and represents the collaborative mixing, e.g.

$$A_{\text{Lab}} \xrightarrow{+\text{patent}|\text{research,legal}} A_{\text{observer}} \quad (66)$$

$$A_{\text{Lab}} \xrightarrow{\pm\text{interact}} A_{\text{services}} \quad (67)$$

$$A_{\text{Lab}} \xrightarrow{-\text{research}} A_{\text{staff}} \quad (68)$$

$$A_{\text{Lab}} \xrightarrow{-\text{legal}} A_{\text{lawyer}} \quad (69)$$

$$A_{\text{staff}} \xrightarrow{+\text{research}} A_{\text{Lab}} \quad (70)$$

$$A_{\text{lawyer}} \xrightarrow{+\text{legal}} A_{\text{Lab}} \quad (71)$$

More generically, with two stages in the process of promise keeping, each experiencing scaling (see figure 9),

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<sup>21</sup>In [32] an explanation based on spanning tree branching processes was postulated, but was not credible.

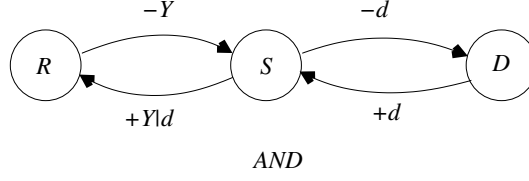


Figure 9: A two stage (long range) dependency has two economies of scale, when fed by a spacetime workflow. The probability of promises kept is multiplicative, like the logical ‘AND’ of the promises.

$$S \xrightarrow{+Y|d} R \quad (72)$$

$$R \xrightarrow{-Y} S \quad (73)$$

$$S \xrightarrow{-d} D \quad (74)$$

$$D \xrightarrow{+d} S \quad (75)$$

the total process picks up two ‘economies of scale’: the delivery of  $Y$  conditionally AND the delivery of the conditional dependence.

$$Y \simeq Y_0 \left( \frac{v}{V_I} \right) N^2 \times \left( \frac{v}{V_{\text{depend}}} \right) N \rightarrow N^{1+2\delta} \simeq N^{\frac{4}{3}} = N^{1.33}. \quad (76)$$

where  $D = 2$  is used for the numerical values. These values accord better with the cited data in [2], and tie in with the story about queueing.

What characterizes this interaction is the high level of specialization required to fulfill the dependencies. If the network is sparse, this is more difficult than if it is dense and diverse. This is the specialization gamble. With specialization comes individual efficiency, but also risk of instability by disconnection from key dependencies [33].

They are a throttle on the process, because their absence could stop it altogether. Hence, we are justified in using the product ‘AND’ for combining the probably values in (76). This is not a

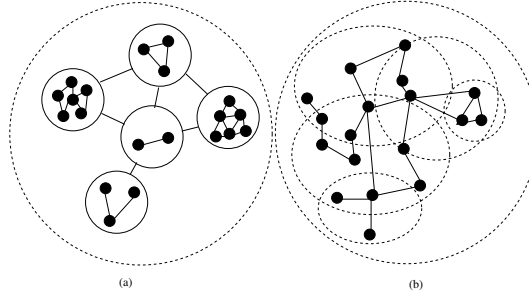


Figure 10: Structural recursion in an ecosystem is not like a branching process of containers (a), but rather the agents overlap with other regions of the same network to access their virtual functions. Thus their outputs are not concealed as interior substructure, but exposed as part of the flat internetwork (b). The result is that a second order recursion picks up a second economy of scale, in turn increasing the superlinearity of the derived output.

hierarchical system interaction, because the services are not necessarily hidden from the long range dynamics by internal components of the superagent ‘lab’ (see figure 10 (b)). But in organizational theory, one normally assumes that all organizations are hierarchically organized (see figure 10 (a)).

4. **Recursive promise dependency.** Let's consider what happens when the ecosystem network is based on a hierarchy of interaction ranges, i.e. promises are made recursively in fully protected shells. The agency produces a service using full community infrastructure, but also has some specialist dependency contained entirely within (see figure 10 (a)).

$$A_{\text{company}} \xrightarrow{+A_{\text{specialist}} \xrightarrow{+\text{solution}} A_{\text{client}}} A_{\text{client}} \quad (77)$$

would be viewed as a recursive operation on the infrastructure, and the economics of scale would apply to both times the (different) infrastructures were used.

$$Y_{\text{patent}} = \frac{N_I^2}{V_{\text{patent}}} \quad (78)$$

$$V_{\text{patent}} = g_{\text{R\&D}} \left( \frac{V_{\text{R\&D}}}{N_{\text{R\&D}}} \right)^{\frac{1}{D}} N_{\text{R\&D}} \quad (79)$$

$$V_{\text{R\&D}} = \left( \frac{V}{N_I} \right)^{\frac{1}{D}} \quad (80)$$

Substituting for  $V_{\text{R\&D}}/N_{\text{R\&D}}$  from the last expression into the former,

$$V_{\text{patent}} = g_{\text{R\&D}} \left( \frac{V}{N_I} \right)^{\frac{1}{D^2}} N_{\text{R\&D}} \quad (81)$$

Inserting this into the output expression

$$Y_{\text{patent}} \simeq N^{1 + \frac{1}{D^2} - \frac{1}{D(D+1)}} \simeq N^{\frac{13}{12}} \simeq N^{1.08}. \quad (82)$$

This value is almost linear, which is what we might expect on a self-contained specialization, since the outside world would not be able to tell the difference between a single agent and a single superagent.

The value is also smaller than case (1) above, not larger, so short range hierarchical scaling cannot explain the anomalously large exponents measured in cities. On the other hand, there are some smaller exponents in this range. It is interesting to examine these measures from the perspective of the promises represented, and their range in the embedding space.

There is a simple prediction here: long range interaction via dependency seems to increase output superlinearity, by compounding economies of scale, i.e. dependency brings strong long range coupling and activates a larger amount of the  $N^2$  mesh. A similar effect can be obtained artificially in [1], by slightly increasing the Hausdorff dimension of the infrastructure  $H > 1$ . This does would correspond to a more pervasive generic infrastructure network, which is an opaque explanation at best. It seems unclear how to justify it.

Short range dependency is basically invisible at larger scales. This observation might help to explain superlinear seen in technological contexts too, through coordination [20], but we have to be careful not to mix together effects that come about due to higher dimensionality, with other mechanisms for increasing the utilization of dependent resources.

### 3.8 Service discovery of dependencies in a network

The ability for agents to discover and match with other agents, that make complementary promises, is the basis of functional scaling, and the semantics of cooperation and innovation. It depends on either

physical or virtual mobility of the agents. Kinetically, agents may follow a random walk, as in ballistic discovery. A second possibility is that cheap intermediaries perform the discovery of specialized roles<sup>22</sup>. We can say that two agents are either

- Physically close.
- Virtually close.

Promise theory also suggests that they may be close in two ways: dynamically close or semantically close (such as when related meanings are similar). The former depends on the length scales of the system (e.g. city) and its structure. The latter can be assumed approximately independent of these scales, because the carriers are very light, cheap, or fast (or, in the case of semantic distance, purely cognitive). If the cost of discovery can be neglected, the cost equation is different: collaboration can be cheaper, and the value of being in close proximity for a particular specialization is reduced<sup>23</sup>.

Directories, maps, and indices [7] are the keys for agents to virtualize discovery of dependencies, and locate one another without physical search in spacetime. Telephone directories map coordinate addresses to names. Yellow pages map coordinates to specializations. Similar specializations are grouped. Shopping malls and industrial estates act as physical directories, where clients can expect to find services in a small volume. Directories may be discovered themselves, or formed by voluntary registration. The value of new bindings overcomes the tendency for similar specializations to repel one another: similar agents may be attracted implicitly (covalently) by the intermediate attraction to clients. Apart from predicting the importance of directories for smart cities, and organizations, this also predicts that the availability of directory information could affect the productivity of a city as a function of size. This effect might not be clearly visible in the ensemble data, since we would need to compare cities at the same value of  $N$ .

The scaling estimates of the city are based on infrastructure where physical motion of the population is based on the cost of traversing some fraction of the length of the city. We can repeat the output calculation to neglect this cost, as is the case in services that do not require physical transport.

- **Physical interaction** (transport/mobility): people move around using transport infrastructure to experience their environment. There is a promise for people to observe their surroundings, for something related to subject  $\tau$ , and this promise is kept fractionally  $\alpha_\tau \in [0, 1]$  during their walk.

Let the linear range of the agent  $A_i$  be some dimensionless fraction per unit time  $rT_{\text{explore}}$  of the size of the city  $V^{1/D}$ , where  $r$  is the speed in units of city size<sup>24</sup>. If the density of impulses per unit length of city region  $\mathcal{I}$  is assumed constant relative to the transport rate (because this is the basis of commerce, i.e. what the city is trying to optimize for people's finite time), then the number of impulses  $I_\tau$  of type  $\tau$ , experienced on such a walk, may be written:

$$I_\tau \propto r_i T_{\text{explore}} V^{1/D} \mathcal{I} \alpha_\tau \quad (83)$$

where  $\alpha_\tau$  is the probability that the person or agent will be receptive to impulses in its environment that are relevant to promises of type  $\tau$ .

Although there is room for inhomogeneous variations in the city regions, in the transport rate  $r$ , and the density of offerings  $\mathcal{I}$ , this will not change the average scaling argument much, as long as  $N$  is large. I make the assumption here that the density of experiences  $\mathcal{I}$  is constant, even though the density of people is related to the city size. This is because the size of a city is constrained by the time rather than the distance (and we are suppressing explicit time).

<sup>22</sup>Electrons play this role in molecular chemistry, or telecommunications in the human realm.

<sup>23</sup>A dependency does not just have to be discovered, but also maintained in a persistent relationship, which accumulates cost over time.

<sup>24</sup>If the person's path is detailed, one could include the Hausdorff dimension of the path and use  $v_i^{H/D}$  as the range, as Bettencourt suggests. I'll ignore this for now, as humans do not tend to move in fractal paths, as his data suggest.

The range will be some fraction of the size of the city, available by transport infrastructure  $V^{1/D}$ . The cost of physically fishing for ideas thus takes the form

$$C \simeq c_Y N_I V^{\frac{1}{D}}, \quad (84)$$

in agreement with the work model of [1]. This applies for physical city interactions, and leads to the same output scaling expression in [1].

$$Y_Y^+ \simeq N^{\frac{2D+1}{D(1+D)}} N_I^{\frac{D^2-D}{D(D+1)}}, \quad (85)$$

$$\simeq N_I^{\frac{7}{6}} \left(1 + \frac{N_0}{N_I}\right)^{\frac{5}{6}}. \quad (86)$$

- **Virtual interaction** (teleport/messaging): people are immobile and send messages to one another, watch entertainment, browse, read, talk, etc. These activities occupy an increasing amount of the time spent by people, not least because it can easily be interleaved with work time. The rate is no longer related to the size of the city, nor is there any obvious boundary to what can be discovered online (since the range of the Internet is even more diverse than a city)<sup>25</sup>. In this case, the impulses are more likely to be related to availability of the fountain itself (e.g. ‘bandwidth’  $B$ ) multiplied the time spent.

$$I_\tau \propto B T_{\text{explore}} \mathcal{I} \propto \tau \quad (87)$$

Discovery of information is the main issue. Before search engines, there were only directories such as white pages (by person) and yellow pages (by promise type). However delivery of what is discovered might still involve spatial constraints, e.g. locating a new car online does not allow it to be teleported to the buyer’s location. However, 3d printing technology might change this, for a class of problems, soon.

Here it is not the locations that matter, but the rate at which impulses are absorbed. Once again, this is constant. When friends, books, or movies are communicating ideas to us, this happens at a rate that depends only on how quickly we can get hold of a stream. How users discover locations online, or by telephone is a separate question. Directories [7], advertisements, and chance all play a role here. The cost of fishing for ideas is thus now independent of the city size. For a community of multiple super-agents, the analogous expression is:

$$C \simeq c N_I B T_{\text{explore}}. \quad (88)$$

If we imagine a community with no other infrastructure except its telecommunications network, and substitute (88) into the detailed balance equation:

$$g_Y \left(\frac{v_Y}{V}\right) N_I^2 \geq c N_I B T_{\text{explore}}. \quad (89)$$

Following through the calculation for the yield estimate identically, we find the scaling is no longer superlinear ( $D = 2, H = 1$ )

$$Y \simeq N^{\frac{H}{D}} N_I^{(1-\frac{H}{D})} \simeq N_I^{\frac{1}{2}} \left(1 + \frac{N_0}{N_I}\right)^{-\frac{1}{2}}. \quad (90)$$

<sup>25</sup>Because telecommunications networks are global, it does not make sense to relate their cost to the size of the city (though this depends on exactly how we model the costs), so the cost depends more on its usage than on its extent. We simply assume that it exists and has sufficient capacity for the  $N_I$  connected residents.

This simple result reflects the intuition that, if we neglect the ‘universal cost’ of telecommunications from the community accounting, then the value generated as a result of collaborative processes is proportional only to the fraction of participants who span the diameter of the city or community. This reproduces the well-known result for mobile ad hoc networking (MANET) [21, 34].

The rate of output based on trawling of ideas and gestation in closed workgroups will be

$$I_{\tau}^{\text{work}} = N_D (c_{\text{phys}} I_{\tau}^{\text{phys}} + c_{\text{virt}} I_{\tau}^{\text{virt}}) \quad (91)$$

and for the entire city of  $N_W$  workplaces:

$$I^{\text{city}} = \sum_{\tau} I_{\tau}^{\text{work}} \simeq N_W \bar{I}^{\text{work}}. \quad (92)$$

The physical channel agrees with Bettencourt’s expression for mixing volume in (4), where  $N_I = N_D N_W$ , and the virtual channel accords with mobile ad hoc networks. It would be interesting to examine communities physically remote from services to see how these predictions match reality.

## 4 Remarks on technological infrastructure and collaborative networks

Cities are just one form of smart adaptive space, for which we now have a new and fascinating insight in the form of statistical scaling data. Remarkably little data are available for computer installations, software development, or smart warehouses, since these reside in the private sector; nonetheless, there may well be insights to gain from studying the relationship between theory and practice, where we can. Some results may have sufficient dynamical similarity to other cases to infer valuable lessons. Although there are qualitative differences between biological organisms and cities, the main features that make computer systems different from cities are the size and timescales involved. Datacentres are still tiny compared to cities (in terms of active agents  $N$ ). Structural changes take place on the order of seconds rather than weeks, and the rates increase as technology advances (see figure 11). The chief lesson we can derive from cities, which might be applied to other smart infrastructure, is the involvement of spacetime relationships in counting, at large  $N$ .

In [6, 7], a generalized abstraction of functional spaces was developed, starting from ‘atomic’ irreducible considerations. By developing the city scaling theory in this framework, one could hope to bridge the gap between disciplines, and promote future analysis of the effect of changing costs and technology. In information technology, dynamical infrastructure, known as cloud computing, offering co-located shared compute, storage and caching, as well as providing facilities for community software development, shared repository models, and finally the pervasive Internet of Things, represent both present and upcoming challenges for infrastructure modelling.

### 4.1 Online communities versus infrastructure clusters

We must be clear about the difference between online communities and physical networks. It has not escaped the notice of [3] that universality of cities implies network scaling in technology-assisted communities, and some data concerning online communities has been examined with interesting and large superlinear behaviour:

IT INFRASTRUCTURE	COMMON	CITY
	$N \simeq 10 - 10^7$	
	Functional modules	
	Long and short range interactions	
$< 10^{-3}s$	NETWORK $\Delta t_{\text{infra}}$	$> 10^3s$
$10^{-6}s - 1s$	AGENTS $\Delta t_{\text{interaction}}$	$1s - 10^4s$
Seconds	TRANSPORT TIME	Hours
software	INNOVATION	hardware
code, services	PROMISES	goods, services
code, memes, habits	EPIDEMIC TRANSMISSION	replicants, memes, habits
Membership	SEMANTIC EDGE	Membership
Latency threshold	DYNAMIC EDGE	Density threshold
Servers, storage	TENANCY UNIT	Homes, offices, storage
Containers, hosts, private nets	PARTITIONING	cubicles, rooms, buildings
Process groups, clusters, datacentres	SUPER-AGENCIES	cubicles, rooms, buildings
$H = 1 - N$	TRAWL PATH DIMENSION $H$	$H = 1-2$
$1 - 3$	EMBEDDING DIMENSION	$D = 2 - 3$

Figure 11: Comparing cities with IT infrastructure.

MEASURE	Average $\beta$	Source
DNS hosts	$1.28 \pm 0.06$	Internet
Total web pages	$2.03 \pm 0.1$	Internet
Active web pages	$1.68 \pm 0.1$	Internet
Contributors	$1.61 \pm 0.1$	Wikipedia
External links	$1.59 \pm 0.2$	Wikipedia
Internal links	$1.21 \pm 0.02$	Wikipedia

Insufficient details are provided to suggest an explanation for the numbers<sup>26</sup>; however, a promise theory approach like the one begun here, may easily play a role in understanding the structural dependences.

Online communities are sociologically interesting, but more practical, from an engineering viewpoint, is to understand how scaling behaviour modulates productivity in smart infrastructure, that mixes physical and virtual mobility. Such ‘smart spaces’ enact a kind of computation to optimize their conditions, and even exhibit some algorithmic qualities. Users interact with hosted services, which are themselves a community of (software) agents, acting as proxies for human intentions. This client interaction behaves like a long-range weak coupling: an autonomous ‘gas’ of visitors<sup>27</sup>, and malls and directories act as catalysts for value generating interactions. Applications and companies, on the other hand, consume unique and shared resources as residents of the infrastructure. They have superagent boundaries around company and functional concerns, which limit internal processes to short-range (and typically strong coupling) interactions, in the manner of a tenancy [7]. Some brief remarks below concern what one might expect to study and find in software agents that are proxies for human intent, and are enhanced with significant automation.

<sup>26</sup> [35] has made some comments on the distribution of community sizes for physical communities.

<sup>27</sup> In [36, 37], I showed how statistical behaviours in computers behave like a gas at equilibrium, with periodic boundary conditions.



## 4.2 The productivity of agents in human-machine systems

Productive output, in agent communities, is driven either by output from agents working independently, in parallel, or from the mesh of interactions between them. It is throttled by the contention for shared resources (serialization or queueing) and the high cost of long range coherence (i.e. equilibration of state, or mutual calibration). These are the main features captured in Gunther’s Universal Scaling [22].

Automation allows individual agents to generate much greater outputs, without the cost of cooperation, so unequal automation might skew the measured performance outputs in empirical studies, making it difficult to compare cities and other systems<sup>28</sup>. When forming a statistical ensemble of systems (cities, organisms, or cloud infrastructure), we have to be sure to compare similar systems. A fundamentally different technology base would undermine the universality of data for individual data points, and could influence the interpretation of the scaling.

As pointed out in section 3.8, the impact of using messenger channels, like telecommunications rather than the physical mixing, is that the process of information and service ‘discovery’ is fundamentally different, altering the costs by removing the imprint of physical scales from the output scaling. Caching becomes a crucial enabler here, because it converts quadratic  $N^2$  costs into linear  $N$  outputs from single agents (thanks to smart behaviour).

For technology infrastructure, the analogue of discovery by wandering around a town browsing store fronts, in the high street or shopping mall, is browsing a database, or directory (e.g. yellow pages, or online shopping catalogue). Information spread by rumour and reputation, in a pre-telecommunications communities, are supplemented by targeted information about recommendations (e.g. Amazon, Google, and Facebook ads and search).

Productivity is not only increased by automation. The cost of transporting and equilibrating goods and services remains. In information technology, larger and larger amounts of data are now stored in physical and data warehouses. Distribution channels, for products and services are mirrored by technologies for data replication, such as Paxos [38] and Raft [39]. These are now being adopted widely, based on the sense that data equilibrium is important to avoid the inconsistency of ‘many worlds’ viewpoints, when interacting with distributed systems. However, the scaling of equilibration software is necessarily poor, since the cost scales superlinearly (typically like  $N^2$ ), to maintain consistent states [20]. Resource ‘latency’ or response time is a key issue, when dependencies are not local. This too can be improved by caching. The cost of transporting data and materials from remote suppliers, either to a city is like the cost of reaching across the planet to rent a virtual machine platform. Ultimately, no community wants to rely on a fragile remote resource. A cheap non-scalable solution is better than an expensive long term one. This has driven the centralization of large cloud installations, where economies of scale can be argued locally, as transport costs and latencies are the responsibility of the clients.

Caching and replication are strategies that decouple long-range dependences, and restore agent autonomy. It is cheap to cache and replicate many technology functions today, offering a kind of ‘smart behaviour’ like a brain to keep interactions local. The mobility of small data is high, as transferring small data is cheap. The mobility of computation has traditionally been low, as computers were large and cumbersome machinery. But that is reversing, thanks to encapsulation methods (superagent technologies), like machine virtualization and so-called ‘containerization’ of software. Moreover, there is computational processing capacity everywhere today, including on smartphones in our pockets, whereas the sheer size of data being collected is growing to impractical levels for transportation. Computation is progressing from being a shared resource, to a ubiquitous agent capability.

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<sup>28</sup>Some notes to this effect were remarked in [10].

### 4.3 Separation of long and short range interactions (modules)

Limiting unnecessary mixing due to long-range or strong-coupling interactions is essential for establishing modular functionality in networks. Modularity promotes specialization, and the quiet isolation for the gestation of lengthy processes like learning and innovation. It makes economic sense, provided the modules do not themselves become bottlenecks. Where humans are coupled strongly, the Dunbar limits for human valency [40] intrude on scalable design, even with the help of automation.

What does this tell us about cloud computing, microservice communities, and the Internet of Things? Suppose the economy of scale were only of the order of 85%, as in cities, this would not be a huge incentive to centralize, if local resources were available too. The economy would have to be offset against effects like latency and long range dependence, equilibration, etc. As our interest in data from embedded sources increases, and we equip environments with smart ecosystems [26], the idea that cloud computing will be localized in large datacentres seems unrealistic. Latency costs suggest a greater delocalization of workforce to eliminate long range interaction.

Managing specialization, or separation of concerns, is a human-technology issue that we are only just starting to grapple with at scale [41]. Modularity has long been a part of system doctrine, but the evolution of so-called ‘microservices’, or small, specialized software services, is now being motivated empirically, by the limitation of Dunbar valency for human agents [40, 42]. Breaking a system into many small specialist parts, each associated with a different human owner, incurs a new cost of service interaction, but this cost may be paid cheaply with automation to alleviate a more expensive human burden: the hard limit of what a human brain can cope with. The density of information services in modern society is now huge, and the complexity of interactions is significantly more than humans can manage unassisted. Technological agents can handle the greater numbers of interactions, but only if they are sufficiently simple that a human collaboration can understand how to design the promises they should keep. Coordination services for replicating siloed resource clusters are being developed, based on the experiences of large industrial providers like Google [43]. These currently rely on long-range data equilibration methods, which is serious throttle on their performance, and must become worse when spacetime dimension plays a role.

Scaling issues pose the question: at what point should systems (cities, datacentres, communities) break up into decentralized regions? Cloud datacentres grew up from the economies of scale that can be achieved through specialized expertise in infrastructure management. We have already witnessed cloud datacentres multiply like power stations, placed at strategic geographic locations, to cover distances. The next logical step is ubiquitous infrastructure. At this point, the economic advantages of physical clustering (indeed cities themselves) may disappear altogether, leaving only the value of centralized meeting places, clubs, conferences, etc for human contact.

### 4.4 Semantic separation versus dynamical separation (silos)

Silos that encapsulate specializations are isolated semantic functions. The appearance of silos in human organizations is often thought of as a negative phenomenon, because it represents an exclusion of outside interests. However, it also has a necessary and positive effect, implying a separation of short range interactions.

This is not the same as separation of dynamical scales. Voluntary partitioning to mitigate contention has clear advantages, if resources permit. Conversely, confusing semantic separation with physical separation may have dynamical consequences. An excellent example of this may be seen from town planning: for a while ‘garden cities’ were a trend, designed with the aim of tidy separation of functions, separated by green spaces. The principle backfired, however, e.g. in Brasilia [44], where different city functions were so distributed that residents have to travel considerable distances to reach town facilities. This led to severe traffic congestion.

When the cost of collaboration between partitioned agencies grows [33], business networks can degenerate and become non-viable, as a larger part of the infrastructure becomes non-contributing. This is a lesson for microservices. IT architects are starting to realize this now, and are opting for ubiquitous ‘hyperconverged’ architectures, i.e. a return to total package servers to reduce latency and congestion. Sometimes regions form by themselves, from dynamical and semantic principles, with partitions based on language, business, geographic centrality, eigenvector centrality [45], etc.

#### **4.5 The mobility of agents in human-machine communities**

In the past, sparse machine resources were immobile, and inputs were brought to the machine for processing. It was cheaper to move inputs to machinery. Today, mobile devices with significant processing capability are commonplace, and encompass computers and manufacturing (aka 3d printing). Particularly, in information infrastructure, the ubiquity of stationary and mobile processing capacity reduces the need for data to be sent over large distances. Conversely, the amount of data expected to come from domestic and industrial sources will grow massively, decreasing relative mobility of data. Mobility of processing resources is now a crucial issue, and is enabled by containment wrapper technologies that form intentional superagent modules around specific functional roles.

The ubiquity and miniaturization of information technology suggests that the role of space dimension may well be a temporary phenomenon, at least for local interactions. Even with ubiquitous information infrastructure, there will still be some role for large datacentres, particularly in the realm of storage archives, since disaster redundancy is one of the critical issues. Similarly, ‘data gravity’ proposes moving the smaller resource to the larger resource, e.g. move computational power to large data instead of assuming that computational power has to be at a fixed location and transporting data.

#### **4.6 Scaling of voluntary cooperation versus imposition**

Garbage collection, sewage, and drainage services are amongst the most important dependencies for a system. They are sinks for pushed output: scaling results for these services would be interesting indeed. One would expect them to be more susceptible to flash floods and long tail behaviours than the corresponding supply networks, as they can only fail catastrophically when a threshold is reached [31]. It would be very interesting to know how the converse of ‘push’ methods, or imposition services, scale with city size: pull on demand, voluntary cooperation are popping up in society to replace mass broadcasting, e.g. video on demand, replacing television broadcasts.

#### **4.7 The rate of change in the system and utilization**

The perception of time has an interesting relationship with long range order. Time is a local concept, as both physicists and distributed systems specialists know only too well. Without long range data synchronization, clocks may not tell the same time, and not just human clocks, but the pulses that drive and clock processes<sup>29</sup>. For example, imagine an orchestra of musicians. The orchestra can play without a conductor if everyone has enough to do all the time, but when every instrument plays a sparse role in the whole, coordination is difficult as the agents are mostly dormant. Multiplexing allows better utilization of a network. When all agents are busy, i.e. utilization is high, process time passes at a faster rate, relative to its surroundings, and coordination becomes easier. Busy agents are ready to receive new inputs and respond with services, suggesting that a highly utilized system could be more efficient, provided that it lies below the queueing instability.

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<sup>29</sup>Each change in an agent is a tick of the network clock, and each partitioned network has its own sense of time, and time only moves forward according to that clock when something happens (this is essentially Einsteinian relativity [6]).

At high speed, the world is dominated by time, and is essentially one dimensional. In a world dominated by space there is multi-dimensionality, and volumes can quickly become relevant in a continuum limit, and time is suppressed by equilibration. The higher dimensionality promotes an increased average connectivity for the promise graph,  $k(N) \simeq N^\delta(D)$ , that depends on the dimension of the embedding space  $D$ . In the complex interaction between space *and* time, all possibilities are on the table.

## 5 Summary

Universality and scaling are powerful notions in science. Having data about the scaling of functional processes, at large and small  $N$ , offers an invaluable insight into what we can expect of technological systems at scale, and their increasing intrusion into human society. Understanding social sciences in terms of laws, analogous to physical law, is an area where progress has been made over the past century. Understanding such patterns in ‘smart’ admixtures of humans and technology seems an even more relevant challenge today [4]. An obvious question becomes: is there something that can be transferred from the study of cities to other network and community systems, e.g.

- Online user communities—human communities in a virtual space.
- Humans-software communities—interacting processes in a virtual space.
- Microservice communities—collaborating agents in a virtual space, interacting with humans.
- Cloud computing—a shared infrastructure community of contending processes.

Datacentres and software systems share a lot of similarities with cities, but there are differences too. Information infrastructure is largely one dimensional, in most locations. In datacentres infrastructure is only just in the process of becoming two and even three dimensional. It is also quite inhomogeneous, and one needs to know when and where dynamical similarities might be exploited to argue dynamical similarity [26]. As  $N$  becomes large, issues of space and time become much more entwined with the more common one-dimensional algorithmic behaviours generally studied in computer science.

Universality reveals emergent laws, on broad scales. However, a fuller understand of systems, whether human cities, smart cities, computers, or any other human structure, is only achieved by describing both dynamics and semantics at micro- and mesoscopic scales. Just as we cannot understand medicine without understanding the functional roles of structures inside organisms, so the functional organs in a city are key to what it does. The universal scaling arguments for urban areas, in [1, 2], are exciting discoveries, as they point to the involvement of spacetime in functional systems, which is still poorly understood in information and computer science.

Datacentres are still tiny compared to cities, but the two are also growing together thanks to the pervasive spread of the Internet. By applying promise theory to expose some short range interactions, we find more possible exponents that match quite well with the anomalous results in citebettencourt1 (see section 3.7). The immediate applications of this approach leads to some basic observations:

- Value creation in communities comes from a mesh of promises whose outcomes funnelled, filtered and powered by serialized processes for supply and harvesting.
- The sparse probabilistic utilization of shared infrastructure allows agents to interleave their efforts and achieve economies of scale. This keeps a fluctuating city in an approximately stable, steady state over short timescales, and admits limited long term growth.
- Specialization of tasks into modular services allows systems to focus their time and capabilities without the cost of context switching. The strategy of specialization also brings fragility: if the

cost of reconnecting the specialists grows too large, the community can fail to keep promises for essential functions [33].

- Increasing autonomy of a system population, due to the availability of personal assistants, and localized capabilities (smartphones, 3d printers, etc) will undermine the need for transport, and the involvement of city scales in many processes.
- Superlinear scaling results from a dependence on exterior specialist agencies. When remote dependencies are involved, staged economies of scale can accumulate bringing superlinear effects, for each remote dependence to the ensemble. If these external dependencies could be redistributed to a purely autonomous attribute of each agent (e.g. when phone boxes are replaced by mobile phones), then scaling would at best be linear, and the artefact of superlinearity would disappear.
- The ability to discover promised information, to mix it and select new combinations, is the basis of innovation and collaboration. A highly discoverable ecosystem is ‘smart’ in the sense that it can adapt and invent. Any space can, in principle, be made smart in this way, if its agencies make the necessary infrastructure promises.

There are two main lessons to take from these notes, that may be surprising to technologists: (i) the behaviour of a system can involve more than one spatial dimension, and (ii) we can measure and describe just how smart and productive functional spaces actually are across a range of scales from local to global.

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## A The value of promises

The value of links in a network depends on the promises they make. The value of a promise is a form of assessment [9] that any agent can make independently. We write an assessment of whether a promise was kept

$$\alpha_i(A_i \xrightarrow{b} A_k) \in [0, 1] \quad (93)$$

to mean the assessment by agent  $A_i$  that the promise from  $A_j$  to  $A_k$  was kept. A valuation is an estimate of what a promise is worth to an agent. This may or may not depend on the assessment of to what extent the promise is kept or not. Every agent assesses on its own calibrated scale. If we want a common currency valuation for all parties, this has to be calibrated by a single agent according to its scale.

The interpretation of value is also an individual judgement that relies on trust, and may be based on accounting of the assessments over time (reputation) [46].

$$\text{My reputation} \propto \sum_{\text{you}} \left( \text{you} \xrightarrow{-b} \text{me} \right) \quad (94)$$

In words, my reputation is proportional to all the number of ‘you’ who (publicly) promise to accept my promised service. Even unilateral promises may have some value:

VALUATION BY $X$	ABOUT PROMISE	REASON FOR VALUE TO $X$
Me	me $\xrightarrow{+b}$ you	An reputation building investment
Me	you $\xrightarrow{+b}$ me	A service that might help me
You	me $\xrightarrow{+b}$ you	A service that might help you
You	you $\xrightarrow{-b}$ me	You need the service now

Cooperative relationships are usually based on conditional assistance [9], and take the form of a conditional equilibrium:

$$S \xrightarrow{+S|M} R \quad (95)$$

$$R \xrightarrow{+M|S} S \quad (96)$$

$$S \xrightarrow{-M} R \quad (97)$$

$$R \xrightarrow{-S} S. \quad (98)$$

in words,  $S$  promises  $R$  a service, if it receives payment  $M$ ; and  $R$  promises to pay  $M$  if it receives service  $S$ . In a network without trust, this is a deadlock. But if any agent trusts the other enough to go first, it is a cyclic generator of a long term relationship. Such relationships imply lasting value, as known from game theory (for a review see [26]). Both agents also promise that they will take (-) what the other is offering unconditionally. This is a signal of trust. Valuations are not necessarily rational to anyone but the agent that makes them, and are unrelated to cost.

The economic value is that something is exchanged, which requires a binding of both + and - promises.

$$S \xrightarrow{+S} R \quad (99)$$

$$R \xrightarrow{-S} S \quad (100)$$



Both agents recognize the value of the other party, so the value exchanged is proportional their assessments that the promises were kept:

$$v_S \propto \alpha_S(S \xrightarrow{+S} R) \alpha_S(R \xrightarrow{-S} S) \quad (101)$$

$$v_R \propto \alpha_R(S \xrightarrow{+S} R) \alpha_R(R \xrightarrow{-S} S). \quad (102)$$

In a community where such transfers are made often and between arbitrary pairs of agents, standards of valuation are equilibrated, and may be exchange in league with a calibration agency (e.g. a bank or government). Thus, in a well-connected community, with a spanning infrastructure, we may posit that the value of a one-way transfer is simply

$$v_C(\Pi_{ij}^S) = c_S \alpha_i \alpha_j, \quad (103)$$

where  $c_S$  is the currency value of a perfect service relationship  $S$ , and  $\alpha_i$  is an impartial assessment of the probability with which  $A_i$  will keep its promise to give or receive  $S$ .

## B Defining the edge of a city or community by membership promises

So far, we've skirted around the issue of what constitutes the city limit, in the definition of a city. The size of the city plays a role in the measurements, and the model in [1] treats the city as an economically inflated balloon, with an edge, so we must understand what constitutes the scope of the city or semantic space. In fact the city is more like a club with membership than a balloon with an edge.

Consider how the network we call a city is reached from beyond. How does it communicate with the outside world? Cities have roads and other infrastructure links in and out of the city (James Blish novels notwithstanding). Should these links be treated as if they were part of the same infrastructure mesh as the city itself? If so, where does the city start and end? The use of dynamical scaling arguments for everything else suggests that there might be a dynamical 'network' answer to this question of boundary conditions, but this is not the case.

Consider a simple thought experiment, in which we start with two separate cities and join them together by increasing the density of connections (see figure 12). As soon as a city is connected to an

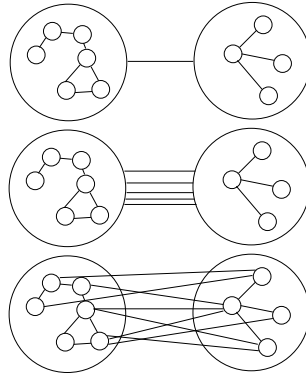


Figure 12: A simple thought experiment: when do two cities merge into a single one? As we add more and more link capacity, when does the scaling change from  $N_1^2 + N_2^2$  to  $(N_1 + N_2)^2$ ? Providing the links are not saturated, it takes only a single link. So where what does the edge mean?

outside network, this extends the internal network. As long as the 'external' network has the capacity

to support the aggregate level of traffic from the population  $N_I$ , then it is no different from the internal infrastructure network, and we have simply extended the boundary of the city.

One proposal might be to look for the presence of a discontinuity in the network capacity, like a threshold event horizon, at which the response time for a network interaction changed:

$$R_{\text{external}} \gg R_{\text{internal}} \quad (104)$$

But this doesn't quite make sense either. Some processes are fast and some are slow, even in an urban centre (a letter posted may take longer to arrive than an office worker takes to process and even reply to it, while a person can take the train across town in half that time)<sup>30</sup>. An ecosystem has a broad mixture of timescales. Only by trying to separating weakly coupled agents can we compare relative timescales for modular components. Moreover, the entirely local gestation or production time is usually the limiting time factor in the economic processes considered here, not the transport or delivery times.

While physically plausible, this is not how cities are defined. They are defined semantically, by labelling of community membership. The simple explanation [6, 9, 29] is that a city is defined to be that collection of agents that mutually promise to be members of the city, and that are accepted as such by the city authorities. In practice, the population must register as residents, and they receive promises of services (including tax collection). One assumes that the transient population of any city is a small correction to this.

The autonomy of any observer counts in making the judgement of community boundary. Each agent can (and will) judge independently whether it considers itself a part of a region or not. This begs the question how collected data define communities, and they really have an edge or not. Can the scaling laws be made to fit any interconnected network? The structural considerations in section 3.7 suggest that, with the right understanding of functional structure, the universalities can be applied properly. These issues have been highlighted in the empirical studies in [10, 13], where it is found that the scaling is distorted by picking the 'wrong boundary' in urban regions.

As we try to apply the ideas to similar networks, such as IT infrastructure and online communities, the role of the physical city as an entity becomes unimportant. It is rather the community that resides and interacts within it that plays the major role of mixing. The details of the infrastructure play a role, but the universality lies in the notion of a community.

## C The phase of a community: mobility of agents, and interaction catalysts

Agents exist a priori in an unbound state, effectively a gas phase, mixing with their changing surroundings. By promising constrained cooperation, they can voluntarily become part of a solid phase, interacting only with fixed neighbours. Mobility of promising agents, in the gas phase, remains important in all functional systems.

There are two kinds of networks in a city:

- Supply or delivery networks which are one-way flows from source to destination. These are mainly branching processes, but may also have simple redundancy.
- Collaboration networks with two-way interactions between communicating agents. The agents can be people, machines, companies, etc.

<sup>30</sup>An impartial approach, based on actual network topology, would be to use the 'Archipelago method', to defining regions of network eigenvector centrality. Using a hill climbing to define natural regions that are seeded on very central nodes leads to well defined regions [45]. The question is whether these have any semantic significance. There are two ways to do it: either based on the shared infrastructure network, or on the virtual business networks that describe the outputs  $Y$ , by defining an effective adjacency matrix based on promise bindings.

Collaborative networks are based in interactions. Chemistry demonstrates that such networks can be realized in two ways: either by using fast messengers (electrons) between slow fixed molecules (like covalence), or by moving faster molecules between slower interaction regions (like ionic), and we may arrange the same methods in larger systems.

- People at fixed locations can use telephone or Internet to send messages (solid state agents).
- People travel between locations, carrying messages with them (agents as a gas).

These methods may be treated as different kinds of network in the model. Promise theory is a chemistry for generalized semantics bindings.

The model proposed by [1] is close to the appearance of a kinetic theory, but the city is not a gas with random motions in this model. It's phase is not defined, because the physical realization of the network is not defined. The movement of the population could be responsible for forming links between agents, i.e. transport via the infrastructure networks (like taxis and subway); or intermediate messenger technologies could be responsible (such as post and Internet). The phase could play a role in the scaling of the infrastructure network in general, constraining its degrees of freedom and range. A solid phase limits the effective dimensionality  $D$ .

Modern cities comprise a fixed infrastructure in a mainly 'solid state', while the agents are sometimes bound in a solid state, and sometimes free as a gas. People move around the city in the subway like water, but increasingly they use messages, like 'covalently' bonded work-molecules rather than transport pipes. Now that distance is less costly, the ease and speed of networking is reflected in the lower density of modern cities, versus older cities where high density enabled ease of meeting, in a kind of primordial soup of mixed intercourse.

Cities are not the only functional networks, of course. Any community could be completely mobile, with no fixed address, such as online communities, coming together only in meeting places that play the role of catalysts. This makes the mixing of skills and promises more efficient<sup>31</sup>. Catalysts for bringing agents together, like social meeting places play an increasingly important role. Open source software is one of the important outputs of the modern world, which happens completely outside cities<sup>32</sup>. In biology, the infrastructure networks are in a fluid state, with functional agents (cells) transported suspension, and their promises advertised by compatibility molecules and receptors on their surfaces, like exterior superagent promises [7], exactly analogous to small businesses. IT networks are mainly solid, or quasi-solid, even when using mobile devices, as the messages move between the agents much faster than the agents move themselves so the motion of devices is negligible for many purposes.

## D Effective power law scaling from Amdahl's and Gunther's law

The Amdahl and Gunther scaling relations are workload scalings, not in the usual form of universal scaling relations. Let's consider how we might derive an approximate power law scaling relation from these. If we want to know how the time fraction (speedup) scales for as a function of the number of processors  $N$ , then we can compare  $N$  with  $\gamma N$ . Then, we can write, for  $\gamma > 1$

$$\frac{T(\gamma N)}{T(N)} = \frac{\sigma + \frac{\pi}{\gamma N}}{\sigma + \frac{\pi}{N}} \quad (105)$$

$$= 1 + \frac{\left(\frac{1}{\gamma} - 1\right) \frac{\pi}{\sigma n}}{1 + \frac{\pi}{\sigma n}}. \quad (106)$$

<sup>31</sup>Curiously, it is also believed in biology that the cooking of food is what made humans an efficient species, supplying energy to fuel our large brains.

<sup>32</sup>Github and other version control repository virtual code libraries function as catalytic meeting places.

Let  $\delta = \frac{D}{D+H}$ , where  $D = 1$  and  $H = \frac{\pi}{\sigma n}$ , then, approximating as a binomial expansion,

$$\frac{T(\gamma N)}{T(N)} = 1 - \left( \frac{\gamma - 1}{\gamma} \right) \delta. \quad (107)$$

$$\simeq \left( 1 - \left( \frac{\gamma - 1}{\gamma} \right) \right)^\delta \dots \quad (108)$$

$$\simeq \gamma^{-\delta}. \quad (109)$$

Thus we have an approximate power law fit for large  $N$  compared to  $\pi/\sigma$ , and we can write

$$T(N) \simeq T_0 N^{-\delta}. \quad (110)$$

i.e. there is a marginal relative economy of scale for small  $N$ , which decays to an essentially scale invariant constant result. If we allow, like Gunther, for the presence of equilibration, or mesh coherence effects, then we could use the form

$$T(N) = \sigma + \frac{\pi}{\gamma N} + \kappa N, \quad (111)$$

where  $\kappa$  represents linear time take to poll each of the worker agents. This is the case where replication and consistency are required. With this extra term, we have

$$\frac{T(\gamma N)}{T(N)} = \frac{\sigma + \frac{\pi}{\gamma N} + \kappa \gamma N}{\sigma + \frac{\pi}{N} + \kappa N} \quad (112)$$

$$= 1 + \frac{\left( \frac{1}{\gamma} - 1 \right) \frac{\pi}{\sigma} + (\gamma - 1) \frac{\kappa}{\sigma} N^2}{1 + \frac{\pi}{\sigma} + \frac{\kappa}{\sigma} N^2}. \quad (113)$$

Now the behaviour doesn't separate cleanly, and there are two regimes of approximate power law scaling, with something more messy in between. Using the same procedure as before, we get an anomalous term for  $\kappa \neq 0$ :

$$\frac{T(\gamma N)}{T(N)} \simeq \left( 1 - \left( \frac{\gamma - 1}{\gamma} \right) \right)^\delta + \frac{\kappa(\gamma - 1)N^2}{\pi + \sigma N + \kappa N^2} \quad (114)$$

$$\simeq \gamma^{-\delta} \quad (\kappa \ll \sigma, N \text{ small}) \quad (115)$$

$$\simeq \gamma \quad (\kappa > 0, N \text{ large}) \quad (116)$$

$$(117)$$

So for large  $N$ , with  $\kappa > 0$ , we have simply

$$T = T_0 N, \quad (118)$$

i.e. the scaling cost becomes linearly worse with increasing size.

When we compare these results to a spacetime scaling, it will become apparent that this takes the approximate form of a scaling law in a one-dimensional spacetime  $D = 1$ , with Hausdorff dimension  $H = \pi/\sigma n < 1$ . This indicates that a serial workflow, with some parallelism, is essentially a one dimensional problem, with some fractal complexity in its trajectory due to parallelism<sup>33</sup>. Interestingly, as the parallelism increases, the duration of the fractal dimensionality shrinks to nothing. Thus the large  $N$  limit for serial processing tends to squeeze the degrees of freedom in the system.

<sup>33</sup>Alternatively, if we think about the problem graph theoretically, we can also say that it behaves like a  $D = N$  dimensional space, and a trajectory with Hausdorff dimension  $H = \pi/\sigma n$ . In a graph, the node degree  $k = N$  is the effective dimension of spacetime at the point [6].